

Arakelov motivic cohomology I

Andreas Holmstrom* and Jakob Scholbach†

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Contents

1	Introduction	2
2	Preliminaries	4
2.1	The stable homotopy category	4
2.2	Beilinson motives	5
2.3	The Riemann-Roch theorem	8
2.4	Deligne cohomology	13
3	The Deligne cohomology spectrum	17
4	Arakelov motivic cohomology	20
4.1	Definition	21
4.2	Functoriality	25
4.3	Purity and an arithmetic Riemann-Roch theorem	27
4.4	Further properties	32

Abstract

This paper introduces a new cohomology theory for schemes of finite type over an arithmetic ring. The main motivation for this Arakelov-theoretic version of motivic cohomology is the conjecture on special values of L -functions and zeta functions formulated by the second author [Sch10]. Taking advantage of the six functors formalism in motivic stable homotopy theory, we establish a number of formal properties, including pullbacks for arbitrary morphisms, pushforwards for projective morphisms between regular schemes, localization sequences, h -descent. We round off the picture with a purity result and a higher arithmetic Riemann-Roch theorem.

*I.H.E.S. Le Bois-Marie, 35 Route de Chartres, F-91440 Bures-sur-Yvette, France, andreas.holmstrom@gmail.com

†Universität Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany, jakob.scholbach@uni-muenster.de

1 Introduction

For varieties over finite fields, we have very good cohomological tools for understanding the associated zeta functions. These tools include ℓ -adic cohomology, explaining the functional equation and the Riemann hypothesis, and Weil-étale cohomology, which allows for precise conjectures and some partial results regarding the “special values”, i.e. the vanishing orders and leading Taylor coefficients at integer values. The conjectural picture for zeta functions of schemes X of finite type over $\mathrm{Spec} \mathbb{Z}$ is less complete. Deninger envisioned a cohomology theory explaining the Riemann hypothesis, and Flach and Morin are developing the Weil-étale cohomology describing special values of zeta functions of regular projective schemes over \mathbb{Z} [Den94, FM10, Mor11].

In [Sch10], the second author proposed a new conjecture, which describes the special values of all zeta functions and L -functions of geometric origin, up to a rational factor. It is essentially a unification of classical conjectures of Beilinson, Soulé and Tate, formulated in terms of the recent Cisinski-Déglise theory of triangulated categories of motives over \mathbb{Z} . This conjecture is formulated in terms of a new cohomology theory for schemes of finite type over \mathbb{Z} , whose properties are described in an axiomatic way in *loc. cit.* The purpose of this paper is to construct this cohomology theory and establish many of its properties.

This cohomology theory, we call it *Arakelov motivic cohomology*, is related to motivic cohomology, roughly in the same way as arithmetic Chow groups relate to ordinary Chow groups or as arithmetic K -theory relates to algebraic K -theory. The key principle for cohomology theories of this type has always been to connect some algebraic data, such as the algebraic K -theory, with an analytical piece of information, chiefly Deligne cohomology, in the sense of long exact sequences featuring the Beilinson regulator map between the two and a third kind of groups measuring the failure of the regulator to be an isomorphism. This was suggested by Deligne and Soulé in the 80s and was initiated by works of Gillet, Roessler, and Soulé who developed a theory of arithmetic Chow groups [GS90b, GS90c, GS90a, Sou92], arithmetic K_0 -theory and an arithmetic Riemann-Roch theorem [Roe99, GRS08]. Burgos and Wang [Bur94, Bur97, BW98] extended some of this to not necessarily projective schemes and gave an explicit representation of the Beilinson regulator. More recently, Goncharov gave a candidate for higher arithmetic Chow groups for complex varieties, Takeda developed higher arithmetic K -theory, while Burgos and Feliu constructed higher arithmetic Chow groups for varieties over arithmetic fields [Gon05, Tak05, BGF]. The analogous amalgamation of topological K -theory and Deligne cohomology of smooth manifolds is known as smooth K -theory [BS09].

In a nutshell, these constructions proceed by representing the regulator as a map of appropriate complexes. Then one defines, say, arithmetic K -theory to be the cohomology of the cone of this map. Doing so, however, requires a good command of the necessary complexes, which so far prevented extending higher arithmetic Chow groups to schemes over \mathbb{Z} and also requires to manually con-

struct homotopies whenever a geometric construction is to be done, for example the pushforward. The idea of this work is to both overcome these hurdles and enhance the scope of these techniques by introducing a *spectrum*, i.e., an object in the stable homotopy category of schemes, representing the sought cohomology theory. Using the abstract machinery of étale descent of spectra, we see that this spectrum does encode the groups we want, and readily gives rise to the features of Arakelov motivic cohomology we are interested in:

Theorem 1.1. *Let S be a regular scheme of finite type over a number field F , a number ring \mathcal{O}_F , \mathbb{R} , or \mathbb{C} . In the stable homotopy category $\mathbf{SH}(S)$ (cf. Section 2.1) there is a ring spectrum H_D representing Deligne cohomology with real coefficients of smooth schemes X/S (Theorem 3.6). This spectrum H_D enjoys a unique $H_{B,S}$ -algebra structure, where $H_{B,S}$ is the spectrum representing motivic cohomology (Theorem 3.6). Essentially, we define the Arakelov motivic cohomology spectrum $\widehat{H}_{B,S}$ as the homotopy fiber of the map*

$$H_{B,S} \xrightarrow{\text{id} \wedge 1_{H_D}} H_{B,S} \wedge H_D.$$

A spectrum \widehat{BGL}_S relating the K -theory spectrum BGL_S and $BGL_S \wedge H_D$ is defined similarly.

We define Arakelov motivic cohomology to be the theory represented by this spectrum, that is to say

$$\widehat{H}^n(M, p) := \text{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(M, \widehat{H}_{B,S}(p)[n])$$

for any $M \in \mathbf{SH}(S)$.

There is a long exact sequence involving Arakelov motivic cohomology, motivic cohomology and Deligne cohomology (Theorem 4.5). This uses that $H_{B,S} \wedge H_D$ is isomorphic to H_D (3.6). Moreover, Arakelov motivic cohomology shares the structural properties known for motivic cohomology, for example a projective bundle formula, a localization sequence, and h -descent (Theorem 4.16). It also has the expected functoriality: pullback for arbitrary morphisms of schemes (or motives, Lemma 4.9) and pushforward along projective maps between regular schemes (Definition and Lemma 4.10).

Last, but not least, we extend the motivic Riemann-Roch theorem given by Riou to arbitrary projective maps between regular schemes (Theorem 2.6)—a statement that is of independent interest. We deduce a purity result and a higher arithmetic Riemann-Roch theorem (Theorem 4.15) for smooth projective morphisms and for projective morphisms between schemes that are smooth over the base ring.

We hope that this work contributes to what might one day grow into a good cohomological picture for ζ -functions over $\text{Spec } \mathbb{Z}$. As mentioned above, Arakelov motivic cohomology is the cohomology theory envisioned in the second author's conjecture. In this conjecture, special L -values for motives M over $\text{Spec } \mathbb{Z}$ are related to the determinant of a *global motivic duality pairing*

$$H_i(M) \times \widehat{H}^{2-i}(M, 1) \rightarrow \mathbb{R}$$

which can be constructed as the composition of morphisms,

$$\mathrm{Hom}(\mathrm{H}_{\mathrm{B}}, M[i]) \times \mathrm{Hom}(M[i], \widehat{\mathrm{H}}_{\mathrm{B}}(1)[2]) \rightarrow \mathrm{Hom}(\mathrm{H}_{\mathrm{B}}, \widehat{\mathrm{H}}_{\mathrm{B}}(1)[2]) = \mathbb{R}.$$

More details on this pairing, as well as the comparison between our construction and the theories of arithmetic K -theory and arithmetic Chow groups, will appear elsewhere.

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2 Preliminaries

In this section, we provide the motivic framework that we are going to work with in Sections 3 and 4: we recall the construction of the stable homotopy category $\mathbf{SH}(S)$ and some properties of the Cisinski-Déglise triangulated category of motives. In Section 2.3, we generalize Riou's formulation of the Riemann-Roch theorem to regular projective morphisms. This will then be used to derive a higher arithmetic Riemann-Roch theorem (4.15). Finally, we recall the definition and basic properties of Deligne cohomology that are needed in Section 3 to construct a spectrum representing Deligne cohomology.

2.1 The stable homotopy category

This section sets the notation and recalls some results pertaining to the homotopy theory of schemes due to Morel and Voevodsky [MV99].

Let S be a Noetherian scheme. We only use schemes which are of finite type over \mathbb{Z} , \mathbb{Q} , or \mathbb{R} . Unless explicitly mentioned otherwise, all morphisms of schemes are understood to be separated and of finite type. Let \mathbf{Sm}/S be the category of smooth schemes over S . The category of presheaves of pointed sets on this category is denoted $\mathbf{PSh}_{\bullet} := \mathbf{PSh}_{\bullet}(\mathbf{Sm}/S)$. We often regard a scheme $X \in \mathbf{Sm}/S$ as the presheaf (of sets) represented by X , and we write $X_+ := X \sqcup \{*\}$ for the associated pointed version. The projective line \mathbb{P}_S^1 is always viewed as pointed by ∞ . The prefix $\Delta^{\mathrm{op}}-$ indicates simplicial objects in a category. The simplicial n -sphere is denoted S^n , this should not cause confusion with the base scheme S .

We consider the *pointwise* and the *motivic model structure* on the category $\Delta^{\mathrm{op}}(\mathbf{PSh}_{\bullet})$ [Jar00, Section 1.1.]. The corresponding homotopy categories will be denoted by $\mathbf{Ho}_{\mathrm{sect}, \bullet}$ and \mathbf{Ho}_{\bullet} , respectively. Recall that the projection $U \times \mathbb{A}^1 \rightarrow U$ is an isomorphism in \mathbf{Ho}_{\bullet} for all $U \in \mathbf{Sm}/S$ and that the Nisnevich topology is included in the motivic model structure as well. The identity functor is a Quillen adjunction with respect to these two model structures.

The category $\mathbf{Spt} := \mathbf{Spt}^{\mathbb{P}^1}(\Delta^{\mathrm{op}}\mathbf{PSh}_{\bullet}(\mathbf{Sm}/S))$ consists of symmetric \mathbb{P}^1_S -spectra, that is, sequences $E = (E_n)_{n \geq 0}$ of simplicial presheaves which are equipped with an action of the symmetric group S_n and S_n -equivariant bonding maps $\mathbb{P}^1 \wedge E_n \rightarrow E_{n+1}$ (and the obvious morphisms). The functor $\Sigma_{\mathbb{P}^1}^{\infty} : \Delta^{\mathrm{op}}(\mathbf{PSh}_{\bullet}) \ni F \mapsto ((\mathbb{P}^1)^{\wedge n} \wedge F)_{n \geq 0}$ (bonding maps are identity maps, S_n acts by permuting the factors \mathbb{P}^1) is left adjoint to $\Omega^{\infty} : (E_n) \mapsto E_0$. Often, we will not distinguish between a simplicial presheaf F and $\Sigma_{\mathbb{P}^1}^{\infty}(F)$.

The category \mathbf{Spt} is endowed with the *stable model structure* [Jar00, Theorems 2.9, 4.15]. The corresponding homotopy category is denoted \mathbf{SH} (or $\mathbf{SH}(S)$) and referred to as the *stable homotopy category* of smooth schemes over S . The pair $(\Sigma_{\mathbb{P}^1}^{\infty}, \Omega^{\infty})$ is a Quillen adjunction with respect to the motivic model structure on $\Delta^{\mathrm{op}}\mathbf{PSh}_{\bullet}$ and the stable model structures on \mathbf{Spt} . We sum up this discussion by saying that there are adjunctions of homotopy categories

$$\mathbf{Ho}_{\mathrm{sect}, \bullet} \rightleftarrows \mathbf{Ho}_{\bullet} \rightleftarrows \mathbf{SH}. \quad (1)$$

The stable homotopy categories are triangulated categories. We will use both the notation $M[p]$ and $M \wedge (S^1)^{\wedge p}$, $p \in \mathbb{Z}$ for the shift functor. Moreover, in $\mathbf{Ho}(S)$, there is an isomorphism $\mathbb{P}_S^1 \cong S^1 \wedge (\mathbb{G}_{m,S}, 1)$. Thus, in $\mathbf{SH}(S)$, wedging with $\mathbb{G}_{m,S}$ is invertible, as well, and we write $M(p)$ for $M \wedge (\mathbb{G}_{m,S})^{\wedge p}[-p]$, $p \in \mathbb{Z}$ for the *Tate twist*. For brevity, we also put

$$M\{p\} := M(p)[2p].$$

For any additive category \mathcal{C} , we write $\mathcal{C}_{\mathbb{Q}}$ for the category with the same objects, but Hom-groups tensored with \mathbb{Q} . Given an object $C \in \mathcal{C}$, we occasionally write $C_{\mathbb{Q}}$ to indicate that same object, seen in $\mathcal{C}_{\mathbb{Q}}$. In particular, we shall use $\mathbf{SH}(S)_{\mathbb{Q}}$. Wherever convenient, we use the equivalence of this category with $\mathbf{D}_{\mathbb{A}^1}(S, \mathbb{Q})$, the homotopy category of symmetric \mathbb{P}^1 -spectra of complexes of Nisnevich sheaves of \mathbb{Q} -vector spaces (with the Tate twist and \mathbb{A}^1 inverted) [CD09, 5.3.22, 5.3.37].

Given a morphism $f : T \rightarrow S$, the stable homotopy categories are connected by adjunctions:

$$f^* : \mathbf{SH}(S) \rightleftarrows \mathbf{SH}(T) : f_*, \quad (2)$$

$$f_! : \mathbf{SH}(T) \rightleftarrows \mathbf{SH}(S) : f^!, \quad (3)$$

$$f_{\sharp} : \mathbf{SH}(T) \rightleftarrows \mathbf{SH}(S) : f^*. \quad (4)$$

For the last adjunction, f is required to be smooth. (2) also applies to morphisms which are not necessarily of finite type ([Ayo07], see also [CD09, 1.1.11, 1.1.13; 2.4.4., 2.4.10]).

2.2 Beilinson motives

Let S be a Noetherian scheme of finite dimension. The key to Beilinson motives (in the sense of Cisinski and Déglise) is the motivic cohomology spectrum

$H_{B,S}$ due to Riou [Rio07, IV.46, IV.72]. There is an object $BGL_S \in \mathbf{SH}(S)$ representing algebraic K -theory in the sense that

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, BGL_S) = K_n(X) \quad (5)$$

for any regular scheme S and any smooth scheme X/S , functorially (with respect to pullback) in X . The \mathbb{Q} -localization $BGL_{S,\mathbb{Q}}$ decomposes as

$$BGL_{S,\mathbb{Q}} = \bigoplus_{p \in \mathbb{Z}} BGL_S^{(p)}$$

such that the pieces $BGL_S^{(p)}$ represent the graded pieces of the γ -filtration on K -theory:

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, BGL_S^{(p)}) \cong \mathrm{gr}_\gamma^p K_n(X)_\mathbb{Q}. \quad (6)$$

The *Beilinson motivic cohomology spectrum* H_B is defined by

$$H_{B,S} := BGL_S^{(0)}. \quad (7)$$

The parts of the K -theory spectrum are related by periodicity isomorphisms

$$BGL_S^{(p)} = H_{B,S}\{p\}. \quad (8)$$

For any map $f : T \rightarrow S$, not necessarily of finite type, there are natural isomorphisms

$$f^* BGL_S = BGL_T, \quad f^* H_{B,S} = H_{B,T}. \quad (9)$$

The following definition and facts are due to Cisinski and Déglise [CD09, Sections 12.3, 13.2]. By a result of Röndigs, Spitzweck and Ostvaer [RSØ10], $BGL_S \in \mathbf{SH}(S)$ is weakly equivalent to a certain cofibrant strict ring spectrum BGL'_S , that is to say a monoid object in the underlying model category $\mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$. In the same vein, $H_{B,S}$ can be represented by a strict commutative monoid object $H'_{B,S}$. The model structures on the subcategory of $\mathbf{Spt}^{\mathbb{P}^1}$ of BGL'_S - and $H'_{B,S}$ -modules are endowed with model structures such that the forgetful functor is Quillen right adjoint to smashing with BGL'_S and $H'_{B,S}$, respectively. The homotopy categories are denoted $\mathbf{DM}_{BGL}(S)$ and $\mathbf{DM}_B(S)$, respectively. Objects in $\mathbf{DM}_B(S)$ will be referred to as *motives* over S . We have adjunctions

$$- \wedge BGL_S : \mathbf{SH}(S) \rightleftarrows \mathbf{DM}_{BGL}(S) : \text{forget} \quad (10)$$

$$- \wedge H_{B,S} : \mathbf{SH}(S)_\mathbb{Q} \rightleftarrows \mathbf{DM}_B(S) : \text{forget}. \quad (11)$$

There is a canonical functor from the localization of $\mathbf{SH}(S)_\mathbb{Q}$ by all H_B -acyclic objects E (i.e., those satisfying $E \otimes H_{B,S} = 0$) to $\mathbf{DM}_B(S)$. This functor is an equivalence of categories, which shows that the above definition is independent of the choice of $H'_{B,S}$. This also has the consequence that the forgetful functor $\mathbf{DM}_B(S) \rightarrow \mathbf{SH}(S)_\mathbb{Q}$ is fully faithful, which will be used in Section 4.1. All these facts stem from the miraculous fact that the multiplication map $H_B \wedge H_B \rightarrow H_B$ is an isomorphism.

Motivic cohomology of any object M in $\mathbf{SH}(S)_{\mathbb{Q}}$ is defined as

$$H^n(M, p) := \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(M, H_{\mathbb{B}}(p)[n]) \stackrel{(11)}{=} \mathrm{Hom}_{\mathbf{DM}_{\mathbb{B}}(S)}(M \wedge H_{\mathbb{B}, S}, H_{\mathbb{B}, S}(p)[n]). \quad (12)$$

The adjunctions (10), (11) are morphisms of *motivic categories* [CD09, Def. 2.4.2], which means in particular that the functors $f_{\sharp}, f_*, f^*, f_!$ and $f^!$ of (2), (3), (4) on $\mathbf{SH}(-)$ can be extended to ones on $\mathbf{DM}_{\mathrm{BGL}}(-)$ and $\mathbf{DM}_{\mathbb{B}}(-)$ in a way that is compatible with these adjunctions. For $\mathbf{DM}_{\mathbb{B}}(S)$ this can be rephrased by saying that these functors preserve the subcategories $\mathbf{DM}_{\mathbb{B}}(-) \subset \mathbf{SH}(-)_{\mathbb{Q}}$.

For any smooth quasi-projective morphism $f : X \rightarrow Y$ of constant relative dimension n and any $M \in \mathbf{SH}(Y)$ (or $\mathbf{DM}_{\mathrm{BGL}}(Y)$, $\mathbf{DM}_{\mathbb{B}}(Y)$), we have the *relative purity* isomorphism (functorial in M and f)

$$f^!M \cong f^*M\{n\}. \quad (13)$$

For example, $f^!H_{\mathbb{B}, Y} \cong H_{\mathbb{B}, X}\{n\}$. This is due to Ayoub, see e.g. [CD09, 2.4.21].

For any closed immersion $i : X \rightarrow Y$ between two regular schemes X and Y with constant relative codimension n , there are *absolute purity* isomorphisms [CD09, 12.6.3, 13.4.1]

$$i^!H_{\mathbb{B}, Y} \cong H_{\mathbb{B}, X}\{-n\}, \quad i^!BGL_Y \cong BGL_X. \quad (14)$$

Definition 2.1. Let $f : X \rightarrow S$ be any map of finite type. We define the *motive* of X over S to be

$$M(X) := M_S(X) := f_!f^!H_{\mathbb{B}, S} \in \mathbf{DM}_{\mathbb{B}}(S).$$

Remark 2.2. In [CD09, 1.1.33] the motive of a smooth scheme $f : X \rightarrow S$ is defined as $f_{\sharp}f^*H_{\mathbb{B}, S}$. These two definitions agree up to functorial isomorphism: we can assume that f is of constant relative dimension d . By relative purity, the functors $f^!$ and $f^*\{d\}$ are isomorphic. Thus their left adjoints, namely $f_!$ and $f_{\sharp}\{-d\}$ agree, too. Therefore, $f_!f^!H_{\mathbb{B}, S} = f_!f^*H_{\mathbb{B}, S}\{d\} = f_{\sharp}f^*H_{\mathbb{B}, S}$.

Definition 2.3. A map $f : X \rightarrow Y$ of S -schemes is a *locally complete intersection (l.c.i.) morphism* if both X and Y are regular and, for simplicity of notation, of constant dimension and if

$$f = p \circ i : X \xrightarrow{i} X' \xrightarrow{p} Y$$

where i is a closed immersion and p is smooth. Note that this implies that X' is regular. If there is such a factorization with $p : X' = \mathbb{P}_Y^n \rightarrow Y$ the projection, we call f a *regular projective map*.

We shall write $\dim f := \dim X - \dim Y$ for any map $f : X \rightarrow Y$ of finite-dimensional schemes.

Example 2.4. Let $f = p \circ i$ be an l.c.i. morphism. Absolute purity for i and relative purity for p give rise to isomorphisms

$$f^!H_{\mathbb{B}, S} \cong f^*H_{\mathbb{B}, S}\{\dim(f)\}, \quad f^!BGL_S \cong f^*BGL_S.$$

Let $f : X \rightarrow Y$ be a projective regular map. Recall the *trace map* in $\mathbf{SH}(Y)$

$$\mathrm{tr}_f^{\mathrm{BGL}} : f_* \mathrm{BGL}_X = p_* i_* i^* \mathrm{BGL}_{X'} \xrightarrow{(14)} p_* \mathrm{BGL}_{X'} \rightarrow \mathrm{BGL}_Y \quad (15)$$

constructed in [CD09, 12.7.3]. This is not an abuse of notation insofar as $\mathrm{tr}_f^{\mathrm{BGL}}$ is independent of the choice of the factorization. This is shown by adapting [Dég08, Lemma 5.11] to the case where all schemes in question are merely regular.

The trace map for $H_{\mathbb{B}}$ is defined as the composition

$$\mathrm{tr}_f^{\mathbb{B}} : f_* f^* H_{\mathbb{B}, Y} \{ \dim f \} \rightarrow f_* f^* \mathrm{BGL}_{\mathbb{Q}, Y} \xrightarrow{\mathrm{tr}_f^{\mathrm{BGL}}} \mathrm{BGL}_{\mathbb{Q}, Y} \rightarrow H_{\mathbb{B}, Y}. \quad (16)$$

In case $f = i$, this is the definition of [CD09, Section 13.4].

Given another regular projective map g , the composition $g \circ f$ is also of this type. The trace maps are functorial: the composition

$$f^* g^* \mathrm{BGL} \xrightarrow{\mathrm{tr}_{g \circ f}^{\mathrm{BGL}}} f^! g^* \mathrm{BGL} \xrightarrow{f^! \mathrm{tr}_g^{\mathrm{BGL}}} f^! g^! \mathrm{BGL}$$

agrees with $\mathrm{tr}_{g \circ f}^{\mathrm{BGL}}$ and similarly with $\mathrm{tr}_?^{\mathbb{B}}$. This can be deduced from the independence of the factorization, cf. [Dég08, Prop. 5.14].

By construction, for any smooth map $f : Y' \rightarrow Y$, the induced map $\mathrm{Hom}(f_{\sharp} f^* S^0, \mathrm{tr}_f^{\mathrm{BGL}}[-n]) : K_n(X') \rightarrow K_n(Y')$ is the K -theoretic pushforward along $f' : X' := X \times_Y Y' \rightarrow Y'$. Similarly, $\mathrm{Hom}(f_{\sharp} f^* S^0, \mathrm{tr}_f^{\mathbb{B}}[-n](p))$ is the pushforward $K_n(X')_{\mathbb{Q}}^{(p)} \rightarrow K_n(Y')_{\mathbb{Q}}^{(p)}$. Indeed, the pushforward on the Adams graded pieces of K -theory is defined as the induced map of the graded homomorphism f'_* on K -theory [FL85, V.6.4]. The adjoint maps

$$\mathrm{BGL}_X = f^* \mathrm{BGL}_Y \rightarrow f^! \mathrm{BGL}_Y, \quad f_* f^* \mathrm{BGL}_Y \rightarrow \mathrm{BGL}_Y$$

will also be denoted $\mathrm{tr}_f^{\mathrm{BGL}}$ and similarly with $\mathrm{tr}_f^{\mathbb{B}}$.

2.3 The Riemann-Roch theorem

In this section we establish two results which have consequences for Arakelov motivic cohomology. The first is a version of smooth base change, and the second, more interesting statement, is a generalization of Riou's motivic Riemann-Roch theorem to projective maps between regular schemes.

The following proposition rephrases the classical smooth base-change formula for K -theory (or, similarly, motivic cohomology), i.e., $f^* g_* = g'_* f'^* : K_0(X) \rightarrow K_0(Y')$, in terms of the agreement of the two morphisms in $\mathbf{SH}(Y)$ inducing these maps:

Proposition 2.5. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow f' & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

be a cartesian square, where f is smooth and g is a regular projective map (therefore, so is g'). Then, the two morphisms in $\mathbf{SH}(Y)$

$$f_{\#}f^*B \xrightarrow{u} g_!g^*f_{\#}f^*B \xrightarrow{(Ex_{\#}^*)^{-1}} g_!f'_{\#}g'^*f^*B \xrightarrow{\mathrm{tr}_{g'}^B} g_!f'_{\#}g'^!f^*B \xrightarrow{Ex^{!*}} g_!f'_{\#}f'^*g^!B \xrightarrow{c} g_!g^!B$$

and

$$f_{\#}f^*B \xrightarrow{c} B \xrightarrow{u} g_!g^*B \xrightarrow{\mathrm{tr}_g^B} g_!g^!B.$$

agree. Here, we have written B for BGL . Moreover, $Ex_{\#}^*$ is the smooth base-change isomorphism [CD09, 1.1.6] and the natural isomorphism $Ex^{!*} : g^!f^* = f'^*g^!$ is obtained by the exchange isomorphism of the left adjoints, $Ex_{\#*} : f_{\#}g'_* = g_*f'_{\#}$ [CD09, 1.1.14]. Throughout, we write c and u for counits and units of adjunctions, respectively.

The analogous statement for H_B (and tr^B) instead of BGL also holds true.

Proof: Using the fact $f'^*\mathrm{tr}_g^B = \mathrm{tr}_{g'}^B$ [CD09, 12.5.5(3) and 12.7.2], or more precisely, the commutativity of

$$\begin{array}{ccc} f'^*g^*B & \xrightarrow{f'^*\mathrm{tr}_g^B} & f'^*g^!B \\ \parallel & & \cong \downarrow Ex^{!*} \\ g'^*f^*B & \xrightarrow{\mathrm{tr}_{g'}^B} & g'^!f^*B, \end{array}$$

we are reduced to proving that in the big solid rectangle below, the morphism given by the bottom vertical row agrees with the one given by the rest of the rectangle.

$$\begin{array}{ccccccc} g_!f'_{\#}g'^*f^*B & \xlongequal{\quad} & g_!f'_{\#}f'^*g^*B & \xrightarrow{\mathrm{tr}_g^B} & g_!f'_{\#}f'^*g^!B & \xrightarrow{Ex^{!*}} & g_!f'_{\#}g'^!f^*B \\ \downarrow u & & \uparrow & & \downarrow \text{dotted} & & \downarrow Ex^{!*} \\ g_!f'_{\#}g'^*f^*f_{\#}f^*B & & & & & & g_!f'_{\#}f'^*g^!B \\ \parallel & \text{dotted } \nearrow c & & & \text{dotted } \searrow id & & \downarrow \\ g_!f'_{\#}f'^*g^*f_{\#}f^*B & & & & & & g_!f'_{\#}f'^*g^!B \\ \downarrow c & & \text{dotted } \nearrow c & & \text{dotted } \searrow c & & \downarrow c \\ g_!g^*f_{\#}f^*B & & & & & & g_!g^!B \\ \uparrow u & & \text{dotted } \nearrow c & & \text{dotted } \searrow c & & \downarrow c \\ \boxed{f_{\#}f^*B} & \xrightarrow{c} & B & \xrightarrow{u} & g_!g^*B & \xrightarrow{\mathrm{tr}_g^B} & \boxed{g_!g^!B} \end{array}$$

(I) (II) (III) (IV) (V)

The agreement between these two morphisms follows from the following elementary observations, where we have added a number of dotted arrows to cut the

rectangle into five smaller diagrams. The diagram (I) commutes by naturality of the unit of $g^* \rightleftarrows g_*$, applied to the morphism $f_{\sharp} f^* B \rightarrow B$. The diagram (II) commutes since $Ex^{!*}$ and $Ex^{*!}$ are mutually inverse isomorphisms. The diagram (III) commutes by naturality of the counit of $f'_{\sharp} \rightleftarrows f'^*$ applied to the morphism $g^* f_{\sharp} f^* B \rightarrow g^* B$. In the diagram (IV), the arrow obtained by starting in the top left corner and going all the way around, agrees with the identity arrow. Indeed, for any pair (L, R) of adjoint functors, the composition $R \rightarrow RLR \rightarrow R$, given by the unit followed by the counit, is the identity. Finally, the diagram (V) commutes by naturality of the counit of $f'_{\sharp} \rightleftarrows f'^*$ applied to the trace map tr_g^B .

The statement for H_B follows from the one for BGL and the definition of tr^B . \square

We now turn to a motivic Riemann-Roch theorem, which will imply an arithmetic Riemann-Roch theorem for Arakelov motivic cohomology (Theorem 4.15). It generalizes the statement given by Riou for smooth morphisms [Rio09, Theorem 6.3.1] to regular projective maps. Independently, F. Déglise has obtained a similar result [Dég11]. Recall the *virtual tangent bundle* of a regular projective map $f = p \circ i : X \xrightarrow{i} X' \xrightarrow{p} Y$, $T_f := i^* T_p - C_{X/X'} \in K_0(X)$ (see e.g. [FL85, V.7]). Here $T_p := \Omega_{X'/Y}^\vee$ is the tangent bundle of p and $C_{X/X'} := (I/I^2)^\vee$ is the conormal sheaf associated the ideal I defining i . As an element of $K_0(X)$, T_f does not depend on the factorization. Its Todd class $\mathrm{Td}(T_f)$ is seen as an endomorphism of $\oplus_{p \in \mathbb{Z}} H_{B,X}\{p\}$ via

$$\mathrm{Td}(T_f) \in \oplus_{p \in \mathbb{Z}} K_0(X)_{\mathbb{Q}}^{(p)} = \mathrm{End}_{\mathbf{DM}_{BGL}(X)_{\mathbb{Q}}}(\oplus_{p \in \mathbb{Z}} H_{B,X}\{p\}). \quad (17)$$

Theorem 2.6. (*Riemann-Roch*) *Let $f : X \rightarrow Y$ be a regular projective map. The following diagram is a commutative diagram in $\mathbf{SH}(Y)_{\mathbb{Q}}$ (or, equivalently, in $\mathbf{DM}_B(Y)$):*

$$\begin{array}{ccc} f_* f^* BGL_{\mathbb{Q},Y} & \xrightarrow{\mathrm{tr}_f^{BGL}} & BGL_{\mathbb{Q},Y} \\ \mathrm{ch} \downarrow \cong & & \mathrm{ch} \downarrow \cong \\ f_* f^* B_Y & \xrightarrow{f_* \mathrm{Td}(T_f)} f_* f^* B_Y \xrightarrow{\mathrm{tr}_f^B} & B_Y. \end{array}$$

Here, B_Y is a shorthand for $\oplus_{p \in \mathbb{Z}} H_{B,Y}\{p\}$.

Proof: The statement is easily seen to be stable under composition of regular projective maps so it suffices to treat the cases $f = p : \mathbb{P}_Y^n \rightarrow Y$ and $f = i : X \rightarrow \mathbb{P}_Y^n$ separately. The former case has been shown by Riou, so we can assume $f : X \rightarrow Y$ is a closed embedding of regular schemes. The classical Riemann-Roch theorem says that the map

$$K_0(X)_{\mathbb{Q}} \rightarrow \oplus_p K_0(Y)_{\mathbb{Q}}^{(p)}, x \mapsto \mathrm{ch} f_*(x) - f_*(\mathrm{Td}(T_f) \cup \mathrm{ch}(x))$$

vanishes. Viewing x as an element of $\mathrm{Hom}_{\mathbf{SH}(Y)_{\mathbb{Q}}}(S^0, f_* f^* \mathrm{BGL}_{\mathbb{Q}, Y})$, this can be rephrased by saying that $x \mapsto \alpha_f \circ x$ is zero, where

$$\alpha_f := \mathrm{ch}_X \circ \mathrm{tr}_f^{\mathrm{BGL}} - \mathrm{tr}_f^{\mathrm{B}} \circ f_* \mathrm{Td}(T_f) \circ f_* f^* \mathrm{ch}_Y \in \mathrm{Hom}(f_* f^* \mathrm{BGL}_{\mathbb{Q}, Y}, \mathrm{B}_Y).$$

To show $\alpha_f = 0$, we first reduce to the case where $f : X \rightarrow Y$ has a retraction, that is, a map $p : Y \rightarrow X$ such that $p \circ f = \mathrm{id}_X$. Then, we prove the theorem by reducing it to the classical Riemann-Roch theorem.

For the first step, recall the deformation to the normal bundle [FL85, IV.5]:

$$\begin{array}{ccccccc} \emptyset & \xrightarrow{\quad} & X & \xrightarrow{i_\infty} & \mathbb{P}_X^1 & \xrightarrow{pr} & X \\ & & \nearrow & & \nearrow i_0 & & \downarrow f \\ & & X & & X & & \downarrow F \\ \tilde{Y} & \xrightarrow{\quad} & \tilde{Y} + Y' & \xrightarrow{s+g'} & M & \xrightarrow{\pi} & Y \\ & \nwarrow & \nwarrow & \nwarrow & \nwarrow & & \nwarrow \\ & & Y' & & Y & & \end{array} \quad (18)$$

We have written $M := \mathrm{Bl}_{X \times \infty}(\mathbb{P}_Y^1)$ and $Y' := \mathbb{P}(C_{X/Y} \oplus \mathcal{O}_X)$, $\tilde{Y} := \mathrm{Bl}_X Y$ and $Y' + \tilde{Y}$ for the scheme defined by the sum of the two divisors. All schemes except $Y' + \tilde{Y}$ are regular, all maps except π and pr are closed immersions. The diagram is commutative and every square in it is cartesian. The map f' has a retraction. We show

$$\alpha_{f'} = 0 \Rightarrow \alpha_f = 0$$

by indicating how to replace each argument in [FL85, proof of Theorem II.1.3], which shows $\alpha_{f'} \circ x = 0 \Rightarrow \alpha_f \circ x = 0$ for any x as above, in a manner that is independent of x .

The identity $f_*(x) = f_* i_0^* pr^*(x) = g^* F_* pr^*(x)$ is replaced by the commutativity of the following diagram of maps of (BGL-)motives, where $v := g \circ f = F \circ i_\infty$:

$$\begin{array}{ccc} F_! F^! \mathrm{BGL}_M & \xrightarrow{\mathcal{O}_{\mathbb{P}_X^1} \in K_0(\mathbb{P}_X^1)} & \mathrm{BGL}_M \\ \mathcal{O}_X \in K_0(X) \downarrow & & \downarrow \mathcal{O}_Y \in K_0(Y) \\ v_! v^! \mathrm{BGL}_M & \xrightarrow{\mathcal{O}_X \in K_0(X)} & g_! g^! \mathrm{BGL}_M \end{array}$$

The maps are given by the indicated structural sheaves in $K_0(?)$, via the identifications of Hom-groups in $\mathbf{DM}_{\mathrm{BGL}}(Y)$ with K -theory. For example, the upper horizontal map is the adjoint map to the inverse of the trace map isomorphism $\mathrm{tr}_F^{\mathrm{BGL}} : F^* \mathrm{BGL} \rightarrow F^! \mathrm{BGL}$, which corresponds via absolute purity to $\mathcal{O}_{\mathbb{P}_X^1} \in K_0(\mathbb{P}_X^1) = \mathrm{Hom}_{\mathbf{DM}_{\mathrm{BGL}}(Y)}(F_! F^! \mathrm{BGL}, \mathrm{BGL})$. The composition of the

map given by $\mathcal{O}_{\mathbb{P}_X^1}$ and \mathcal{O}_Y is given by their tensor product (viewed as \mathcal{O}_M -modules), that is, \mathcal{O}_X , so the diagram commutes. The same argument applies to $f'_*(x) = g'^*F_*pr^*(x)$.

The projection formula is [CD09, Theorem 2.4.21(v)]. The identity $g_*(1) = g'_*(1) + s_*(1) \in K_0(M)_{\mathbb{Q}}^{(1)}$ is equivalent to the agreement of the following two elements of $\mathrm{Hom}(\mathrm{H}_{\mathrm{B},M}, \mathrm{H}_{\mathrm{B},M}\{-1\})$:

$$\mathrm{H}_{\mathrm{B},M} \xrightarrow{\mathrm{adj.}} g_*g^*\mathrm{H}_{\mathrm{B},M} \xrightarrow{g_!\mathrm{tr}_g^{\mathrm{B}}} g_!g^!\mathrm{H}_{\mathrm{B},M}\{-1\} \xrightarrow{\mathrm{adj.}} \mathrm{H}_{\mathrm{B},M}\{-1\}$$

and

$$\mathrm{H}_{\mathrm{B},M} \xrightarrow{\mathrm{adj.}} g'_*g'^*\mathrm{H}_{\mathrm{B},M} \oplus s_*s^*\mathrm{H}_{\mathrm{B},M} \xrightarrow{g'_!\mathrm{tr}_{g'}^{\mathrm{B}} \oplus s_!\mathrm{tr}_s^{\mathrm{B}}} g'_!g'^!\mathrm{H}_{\mathrm{B},M}\{-1\} \oplus s_!s^!\mathrm{H}_{\mathrm{B},M}\{-1\} \xrightarrow{\mathrm{adj.}} \mathrm{H}_{\mathrm{B},M}\{-1\}.$$

Finally, the identity $s^*F_*pr^*(x) = 0$ is formulated independently of x using again base-change (and using that the motive of the empty scheme is zero). This finishes the first step.

Thus, we can assume that f has a retraction $p : Y \rightarrow X$. By [Rio09, Section 5, esp. 5.3.6, cf. the proof of 6.1.3.2], the obvious "evaluation" maps $\mathrm{Hom}(\mathrm{BGL}_{X,\mathbb{Q}}, \mathrm{BGL}_{X,\mathbb{Q}})$ injectively to

$$\prod_{i \in \mathbb{Z}, T \in \mathbf{Sm}/X} \mathrm{Hom}_{\mathbb{Q}} \left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, \mathrm{BGL}_{X,\mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, \mathrm{BGL}_{X,\mathbb{Q}}) \right).$$

The outer Hom denotes \mathbb{Q} -linear maps, the inner ones are morphisms in $\mathbf{SH}(X)_{\mathbb{Q}}$. There is an isomorphism $u : f^*\mathrm{BGL}_{Y,\mathbb{Q}} \rightarrow f^!\mathrm{B}_Y$, for example the Chern class followed by the absolute purity isomorphism (Example 2.4). Appending u on both sides, we conclude that the evaluation maps $\mathrm{Hom}(f^*\mathrm{BGL}_{Y,\mathbb{Q}}, f^!\mathrm{B}_Y)$ into

$$\prod_{i,T} \mathrm{Hom}_{\mathbb{Q}} \left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, f^*\mathrm{BGL}_{Y,\mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, f^!\mathrm{B}_Y) \right).$$

For any $T \in \mathbf{Sm}/X$, consider the following cartesian diagram:

$$\begin{array}{ccccc} T & \xrightarrow{f_T} & U & \xrightarrow{p_T} & T \\ \downarrow t & & \downarrow u & & \downarrow t \\ X & \xrightarrow{f} & Y & \xrightarrow{p} & X. \end{array}$$

Recall that $T \in \mathbf{SH}(X)$ is given by $t_{\sharp}t^*S^0$. Here t_{\sharp} is left adjoint to t^* , cf. (4). Thus, the term simplifies to

$$\prod_{i,T} \mathrm{Hom}_{\mathbb{Q}} \left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, t^*f^*\mathrm{BGL}_{Y,\mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, t^*f^!\mathrm{B}_Y) \right).$$

The diagram $X \rightarrow Y \rightarrow X$ is stable with respect to smooth pullback: f_T is also an embedding of regular schemes, p_T is a retract of f_T . Moreover, the trace

map $\mathrm{tr}_f^{\mathrm{BGL}}$ behaves well with respect to smooth pullback, i.e., $t^* \mathrm{tr}_f^{\mathrm{BGL}} = \mathrm{tr}_{f_T}^{\mathrm{BGL}}$ and similarly for $\mathrm{tr}_?^{\mathrm{B}}$, $\mathrm{ch}_?$ and $\mathrm{Td}(T_?)$. Thus, it is sufficient to consider the case $T = X$. That is, we have to show that β_f , the image of α_f in

$$\begin{aligned} & \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{Q}} \left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, f^* \mathrm{BGL}_{Y, \mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, f^! \mathbb{B}_Y) \right) \\ &= \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{Q}} \left(\mathrm{Hom}_{\mathbf{SH}(X)_{\mathbb{Q}}}((\mathbb{P}_X^1)^{\wedge i}, \mathrm{BGL}_{X, \mathbb{Q}}), \mathrm{Hom}_{\mathbf{SH}(Y)_{\mathbb{Q}}}((\mathbb{P}_Y^1)^{\wedge i}, f_* f^! \mathbb{B}_Y) \right) \end{aligned}$$

is zero. The composition

$$\mathrm{Hom}((\mathbb{P}_Y^1)^{\wedge i}, f_! f^* \mathbb{B}_Y) \xrightarrow{\mathrm{tr}_f^{\mathrm{B}}, \cong} \mathrm{Hom}((\mathbb{P}_Y^1)^{\wedge i}, f_! f^! \mathbb{B}_Y) \xrightarrow{\gamma_f} \mathrm{Hom}((\mathbb{P}_Y^1)^{\wedge i}, \mathbb{B}_Y)$$

is the pushforward $f_* : \oplus_{p \in \mathbb{Z}} K_0(X)_{\mathbb{Q}}^{(p)} \rightarrow \oplus K_0(Y)_{\mathbb{Q}}^{(p)}$, which is injective since $p_* f_* = \mathrm{id}$. Thus, the right hand adjunction map γ_f is also injective and it is sufficient to show $\gamma_f \circ \beta_f = 0$. For any $i \in \mathbb{Z}$,

$$\begin{aligned} \gamma_f \circ \beta_f & \stackrel{\text{by def.}}{=} (f_* \circ (- \cup \mathrm{Td}(T_f)) \circ \mathrm{ch}_X) - (\mathrm{ch}_Y \circ f_*) \\ & \stackrel{\text{RR}}{=} 0 \\ & \in \mathrm{Hom}_{\mathbb{Q}} \left(K_0(X)_{\mathbb{Q}}, \oplus K_0(Y)_{\mathbb{Q}}^{(p)} \right) \\ &= \mathrm{Hom}_{\mathbb{Q}} \left(\mathrm{Hom}_{\mathbf{SH}(X)_{\mathbb{Q}}}((\mathbb{P}^1)^{\wedge i}, f^* \mathrm{BGL}_{Y, \mathbb{Q}}), \mathrm{Hom}_{\mathbf{SH}(Y)_{\mathbb{Q}}}((\mathbb{P}^1)^{\wedge i}, \mathbb{B}_Y) \right). \end{aligned}$$

The vanishing labeled RR is the classical Riemann-Roch theorem for f . \square

2.4 Deligne cohomology

Definition 2.7. [GS90a, 3.1.1.] An *arithmetic ring* is a datum $(S, \Sigma, \mathrm{Fr}_{\infty})$, where S is a ring, $\Sigma = \{\sigma_1, \dots, \sigma_n : S \rightarrow \mathbb{C}\}$ is a set of embeddings of S into \mathbb{C} and $\mathrm{Fr}_{\infty} : \mathbb{C}^{\Sigma} \rightarrow \mathbb{C}^{\Sigma}$ is a \mathbb{C} -antilinear involution (called *infinite Frobenius*) such that $\mathrm{Fr}_{\infty} \circ \sigma = \sigma$, where $\sigma = (\sigma_i)_i : S \rightarrow \mathbb{C}^{\Sigma}$. For simplicity, we suppose that $S_{\eta} := S \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Q}$ is a field. If S happens to be a field itself, we refer to it as a *arithmetic field*. For any scheme X over an arithmetic ring S , we write

$$X_{\mathbb{C}} := X \times_{S, \sigma} \mathbb{C}^{\Sigma}$$

and $X(\mathbb{C})$ for the associated complex-analytic space (with its classical topology). We also write $\mathrm{Fr}_{\infty} : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ for the pullback of infinite Frobenius on the base.

The examples to have in mind are the spectra of number rings, number fields, \mathbb{R} or \mathbb{C} , equipped with the usual finite set Σ of complex embeddings and $\mathrm{Fr}_{\infty} : (z_v)_{v \in \Sigma} \mapsto (\overline{z_v})_v$.

We recall the properties of Deligne cohomology that we need in the sequel. In order to construct a spectrum representing Deligne cohomology in Section 3 we recall Burgos' explicit complex whose cohomology groups identify with Deligne cohomology. In the remainder of this subsection, X/S is a smooth scheme (of finite type) over an arithmetic field.

Definition 2.8. [Bur97, Def. 1.2, Thm. 2.6] Let $E^*(X(\mathbb{C}))$ be the following complex:

$$E^*(X(\mathbb{C})) := \varinjlim E_{\overline{X}(\mathbb{C})}^*(\log D(\mathbb{C})), \quad (19)$$

where the colimit is over the (directed) category of smooth compactifications \overline{X} of X such that $D := \overline{X} \setminus X$ is a divisor with normal crossings. The complex $E_{\overline{X}(\mathbb{C})}^*(\log D(\mathbb{C}))$ is the complex of C^∞ -differential forms on $\overline{X}(\mathbb{C})$ that have at most logarithmic poles along the divisor (see *loc. cit.* for details). We write $E^*(X) \subset E^*(X(\mathbb{C}))$ for the subcomplex of elements fixed under the $\overline{\text{Fr}_\infty^*}$ -action. Forms in $E^*(X)$ that are fixed under complex conjugation are referred to as real forms and denoted $E_{\mathbb{R}}^*(X)$. As usual, a twist is written as $E_{\mathbb{R}}^*(X)(p) := (2\pi i)^p E_{\mathbb{R}}^*(X) \subset E^*(X)$. The complex $E^*(X)$ is filtered by

$$F^p E^*(X) := \oplus_{a \geq p, a+b=*} E^{a,b}(X).$$

Let $D^*(X, p)$ be the complex defined by

$$D^n(X, p) := \begin{cases} E_{\mathbb{R}}^{2p+n-1}(X)(p-1) \cap \oplus_{a+b=2p+n-1, a, b < p} E^{a,b}(X) & n < 0 \\ E_{\mathbb{R}}^{2p+n}(X)(p) \cap \oplus_{a+b=2p+n, a, b \geq p} E^{a,b}(X) & n \geq 0 \end{cases}$$

The differential $d_D(x)$, $x \in D^n(X, p)$, is defined as $-\text{proj}(dx)$ ($n < -1$), $-2\partial\bar{\partial}x$ ($n = -1$), and dx ($n \geq 0$). Here d is the standard exterior derivative, and proj denotes the projection onto the space of forms of the appropriate bidegrees. We also set

$$D := \bigoplus_{p \in \mathbb{Z}} D(p).$$

The pullback of differential forms turns D into complexes of presheaves on \mathbf{Sm}/S . *Deligne cohomology* (with real coefficients) of X is defined as

$$H_D^n(X, p) := H^{n-2p}(D(p)(X)).$$

For a scheme X over an arithmetic ring, such that $X_\eta := X \times_S S_\eta$ is smooth (possibly empty), we set $H_D^n(X, p) := H_D^n(X_\eta)$.

Recall that a complex of presheaves $X \mapsto F_*(X)$ on \mathbf{Sm}/S is said to have *étale descent* if for any $X \in \mathbf{Sm}/S$ and any étale map $f : Y \rightarrow X$ the canonical map

$$F_*(X) \rightarrow \text{Tot}(F_*(\dots \rightarrow Y \times_X Y \rightarrow Y))$$

is a quasi-isomorphism. The right hand side is the total complex of F_* applied to the Čech nerve. At least if F is a complex of presheaves of \mathbb{Q} -vector spaces, this is equivalent to the requirement that

$$F_*(X) \rightarrow \text{Tot}(F_*(\mathcal{Y}))$$

is a quasi-isomorphism for any étale hypercover $\mathcal{Y} \rightarrow X$. Indeed the latter is equivalent to F_* satisfying Galois descent (as in (27)) and Nisnevich descent in the sense of hypercovers. The latter is equivalent to the one in the sense of Čech nerves by the Morel-Voevodsky criterion (see e.g. [CD09, 3.3.2, Theorem 3.3.22]).

Theorem 2.9. (i) *The previous definition of Deligne cohomology agrees with the classical one (for which see e.g. [EV88]). In particular, there is a long exact sequence*

$$H_D^n(X, p) \rightarrow H^n(X(\mathbb{C}), \mathbb{R}(p))^{(-1)^p} \rightarrow (H_{dR}^n(X_{\mathbb{C}})/F^p H_{dR}^n(X_{\mathbb{C}}))^{\text{Fr}_{\infty}} \rightarrow H_D^{n+1}(X, p) \quad (20)$$

involving Deligne cohomology, the $(-1)^p$ -eigenspace of the $\overline{\text{Fr}_{\infty}^}$ action on Betti cohomology, and the Fr_{∞} -invariant subspace of de Rham cohomology modulo the Hodge filtration.*

(ii) *The complex $D(p)$ is homotopy invariant in the sense that the projection map $X \times \mathbb{A}^1 \rightarrow X$ induces a quasi-isomorphism $D(\mathbb{A}^1 \times X) \rightarrow D(X)$ for any $X \in \mathbf{Sm}/S$.*

(iii) *There is a functorial first Chern class map*

$$c_1 : \text{Pic}(X) \rightarrow H_D^2(X, 1). \quad (21)$$

(iv) *The complex D is a unital differential bigraded \mathbb{Q} -algebra which is associative and commutative up to homotopy. The product of two sections will be denoted by $a \cdot_D b$. The induced product on Deligne cohomology agrees with the classical product \cup on these groups [EV88, Section 3]. Moreover, for a section $x \in D_0(X)$ satisfying $d_D(x) (= dx) = 0$ and any two sections $y, z \in D_*(X)$, we have*

$$x \cdot_D (y \cdot_D z) = (x \cdot_D y) \cdot_D z \quad (22)$$

and

$$x \cdot_D y = y \cdot_D x. \quad (23)$$

(v) *Let E be a vector bundle of rank r over X . Let $p : P := \mathbf{P}(E) \rightarrow X$ be the projectivization of E with tautological bundle $\mathcal{O}_P(-1)$. Then there is an isomorphism*

$$p^*(-) \cup c_1(\mathcal{O}_P(1))^{\cup i} : \oplus_{i=0}^{r-1} H_D^{n-2i}(X, p-i) \rightarrow H_D^n(P, p). \quad (24)$$

In particular the following Künneth-type formula holds:

$$H_D^n(\mathbb{P}^1 \times X, p) \cong H_D^{n-2}(X, p-1) \oplus H_D^n(X, p). \quad (25)$$

(vi) *The complex of presheaves $D(p)$ satisfies étale descent.*

Proof: (i): This explicit presentation of Deligne cohomology is due to Burgos [Bur97, Prop. 1.3.]. The sequence (20) is a consequence of this and the degeneration of the Hodge to de Rham spectral sequence. See e.g. [EV88, Cor. 2.10]. (ii) follows from (20) and the homotopy invariance of Betti cohomology, de Rham cohomology, and, by functoriality of the Hodge filtration, homotopy

invariance of $F^p H_{\text{dR}}^n(-)$. For (iii), see [BGKK07, Section 5.1.] (or [EV88, Section 7] for the case of a proper variety). (iv) is [Bur97, Theorem 3.3.].¹ For (v), see e.g. [EV88, Prop. 8.5.].

(vi): This statement can be read off the existence of the absolute Hodge realization functor [Hub00, Corollary 2.3.5] (and also seems to be folklore). Since it is crucial for us in Theorem 3.6, we give a proof here. Let

$$\tilde{D}^*(X, p) := \text{cone}(E_{\mathbb{R}}^*(X)(p) \oplus F^p E^*(X) \xrightarrow{(+1, -1)} E^*(X))[-1 + 2p].$$

By [Bur97, Theorem 2.6.], there is a natural (fairly concrete) homotopy equivalence between the complexes of presheaves $\tilde{D}(p)$ and $D(p)$. The descent statement is stable under quasi-isomorphisms of complexes of presheaves and cones of maps of such complexes. Therefore it is sufficient to show descent for the complexes $E_{\mathbb{R}}^*(-)(p)$, $F^p E^*(-)$, $E^*(-)$. Taking invariants of these complex under the Fr_{∞}^* -action is an exact functor, so we can disregard that operation in the sequel. From now on, everything refers to the analytic topology, in particular we just write X for $X(\mathbb{C})$ etc. Let $j : X \rightarrow \overline{X}$ be an open immersion into a smooth compactification such that $D := \overline{X} \setminus X$ is a divisor with normal crossings. The inclusion

$$\Omega_{\overline{X}}^*(\log D) \subset E_X^*(\log D)$$

of holomorphic forms into C^∞ -forms (both with logarithmic poles) yields quasi-isomorphisms of complexes of vector spaces

$$R\Gamma Rj_* \mathbb{C} \rightarrow R\Gamma Rj_* \Omega_X^* \leftarrow R\Gamma \Omega_{\overline{X}}^*(\log D) \rightarrow \Gamma E_X^*(\log D)$$

that are compatible with both the real structure and the Hodge filtration [Bur94, Theorem 2.1.], [Del71, 3.1.7, 3.1.8]. Here $(R)\Gamma$ denotes the (total derived functor of the) global section functor on \overline{X} . The complex $E^*(X)$, whose cohomology is $H^*(X, \mathbb{C})$, is known to satisfy étale descent [Hub00, Prop. 2.1.7]. This also applies to $E_{\mathbb{R}}^*(X)(p)$ instead of $E^*(X)$. (Alternatively for the former, see also [CD07, 3.1.3] for the étale descent of the algebraic de Rham complex Ω_X^* .)

It remains to show the descent for $X \mapsto F^p E^*(X)$. Consider a distinguished square in \mathbf{Sm}/S ,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y, \end{array}$$

i.e., cartesian such that $Y' \rightarrow Y$ is an open immersion, X/Y is étale and induces an isomorphism $(X \setminus X')_{\text{red}} \rightarrow (Y \setminus Y')_{\text{red}}$. Then the sequence

$$H^n(F^p E^*(Y)) \rightarrow H^n(F^p E^*(Y')) \oplus H^n(F^p E^*(X)) \rightarrow H^n(F^p E^*(X')) \rightarrow H^{n+1}(F^p E^*(Y)) \quad (26)$$

¹Actually, the product on $D(X)$ is commutative on the nose. We shall only use the commutativity in the case stated in (23) and the associativity as in (22), cf. Definition and Lemma 3.3.

is exact: first, the direct limit in (19) is exact. Moreover, $H^n(\Gamma(F^p E_{\overline{X}}(\log D)))$ maps injectively into $H^n(\overline{X}, \Omega_{\overline{X}}^*(\log D))$, and the image is precisely the p -th filtration step of the Hodge filtration on $H^n(\overline{X}, \Omega_{\overline{X}}^*(\log D)) = H^n(X, \mathbb{C})$. Similarly for X' etc., so that the exactness of (26) results from the one of the sequence featuring the Betti cohomology groups of Y , $Y' \sqcup X$ and X' , together with the strictness of the Hodge filtration [Del71, Th. 1.2.10]. This shows Nisnevich descent for the Hodge filtration. Secondly, for any scheme X and a Galois cover $Y \rightarrow X$ with group G , the pullback map into the G -invariant subspace

$$H^n(F^p E^*(X)) \rightarrow H^n(F^p E^*(Y)^G) \quad (27)$$

is an isomorphism. Indeed, the similar statement holds for $E^*(-)$ instead of $F^p E^*(-)$. We work with \mathbb{Q} -coefficients, so taking G -invariants is an exact functor, hence $H^n(F^p E^*(Y)^G) = (H^n(F^p E^*(Y)))^G = (F^p H_{\text{dR}}^n(Y))^G = F^p(H_{\text{dR}}^n(Y)^G)$, the last equality by functoriality of the Hodge filtration. Then, again using the strictness of the Hodge filtration, the claim follows. Hence the presheaf $X \mapsto F^p E^*(X)$ has étale descent. \square

3 The Deligne cohomology spectrum

Let S be a smooth scheme (of finite type) over an arithmetic field (Definition 2.7). The aim of this section is to construct a ring spectrum in $\mathbf{SH}(S)$ which represents Deligne cohomology for smooth schemes X over S . The method is a slight variation of the method of Cisinski and Deglise used in [CD07] to construct a spectrum for any mixed Weil cohomology, such as algebraic or analytic de Rham cohomology, Betti cohomology, and (geometric) étale cohomology. The difference compared to their setting is that the Tate twist on Deligne cohomology groups is not an isomorphism of vector spaces.

In this section, all complexes of (presheaves of) abelian groups are considered with homological indexing: the degree of the differential is -1 and $C[1]$ is the complex whose n -th group is C_{n+1} . As usual, any cohomological complex is understood as a homological one by relabeling the indices. In particular, we apply this to $D(p)$, D (Definition 2.8) and let

$$D_n := D^{-n} = \bigoplus_{p \in \mathbb{Z}} D^{-n}(p). \quad (28)$$

In order to have a complex of simplicial presheaves (as opposed to a complex of abelian groups), we use the Dold-Kan-equivalence

$$\mathcal{K} : \mathbf{Com}_{\geq 0}(\mathbf{Ab}) \rightleftarrows \Delta^{\text{op}}(\mathbf{Ab}) : \mathcal{N}$$

between homological complexes concentrated in degrees ≥ 0 and simplicial abelian groups. We write $\tau_{\geq n}$ for the good truncation of a complex.

Definition 3.1. We write

$$\begin{aligned} D_s &:= \mathcal{K}(\tau_{\geq 0} D), \\ D_s(p) &:= \mathcal{K}(\tau_{\geq 0} D(p)). \end{aligned}$$

Via the Alexander-Whitney map, the product on D transfers to a product

$$D_s(p) \wedge D_s(p') \rightarrow D_s(p + p'). \quad (29)$$

Lemma 3.2. *For X smooth over S and any $k \geq 0$, $p \in \mathbb{Z}$ we have:*

$$\mathrm{Hom}_{\mathbf{Ho}_\bullet}(S^k \wedge X_+, D_s(p)) = H_D^{2p-k}(X, p) \quad (30)$$

and similarly for D_s .

Proof: In $\mathbf{Ho}_{\mathrm{sect}, \bullet}$ (cf. Section 2.1 for the notation), the Hom-group reads

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Ho}_{\mathrm{sect}, \bullet}}(S^k \wedge X_+, \mathcal{K}(\tau_{\geq 0}(D))) &= \pi_k \mathcal{K}(\tau_{\geq 0}(D(X))) \\ &= H_k(\tau_{\geq 0}(D(X))) \\ &= \bigoplus_{p \in \mathbb{Z}} H_D^{2p-k}(X, p). \end{aligned}$$

We have used the fact that any simplicial abelian group is a fibrant simplicial set and the identification $\pi_n(A, 0) = H_n(\mathcal{N}(A))$ for any simplicial abelian group.

The presheaf D_s is fibrant with respect to the motivic model structure, since Deligne cohomology satisfies Nisnevich descent and is \mathbb{A}^1 -invariant by Theorem 2.9 (vi) and (ii). Thus the Hom-groups agree when taken in $\mathbf{Ho}_{\mathrm{sect}, \bullet}$ and \mathbf{Ho} , respectively. \square

Definition and Lemma 3.3. *The Deligne cohomology spectrum H_D is the symmetric \mathbb{P}^1 -spectrum consisting of the $D_s(p)$ ($p \geq 0$), equipped with the trivial action of the symmetric group Σ_p . We define the bonding maps to be the composition*

$$\sigma_p : \mathbb{P}_S^1 \wedge D_s(p) \xrightarrow{c^* \wedge \mathrm{id}} D_s(1) \wedge D_s(p) \xrightarrow{(29)} D_s(p+1).$$

Here c^* is the map induced by $c := c_1(\mathcal{O}_{\mathbb{P}^1}(1), FS) \in D_0(1)(\mathbb{P}^1)$, the first Chern form of the bundle $\mathcal{O}(1)$ equipped with the Fubini-Study metric.

We equip H_D with the following monoid structure: the product $\mu : H_D \wedge H_D \rightarrow H_D$ is induced by (29). The unit map $\eta : \Sigma_{\mathbb{P}^1}^\infty S_+ \rightarrow H_D$ is defined in degree zero by the unit of the DGA $D(0)$. In higher degrees, we put

$$\eta_p : (\mathbb{P}^1)^{\wedge p} \xrightarrow{(c^*)^{\wedge p}} D_s(1)^{\wedge p} \xrightarrow{\mu} D_s(p).$$

Equivalently, $\eta_p := \sigma_{p-1} \circ (\mathrm{id}_{\mathbb{P}^1} \wedge \eta_{p-1})$.

This defines a symmetric \mathbb{P}^1 -spectrum in $\Delta^{\mathrm{op}} \mathbf{PSh}(\mathbf{Sm}/S)$. Up to homotopy, that is, as an object of $\mathbf{SH}(S)$, H_D is a ring spectrum.

Proof: Recall that c is a $(1, 1)$ -form which is invariant under Fr_∞^* and under complex conjugation, so c is indeed an element of $D_0(1)(\mathbb{P}^1)$. Its restriction to the point ∞ is zero for dimension reasons, so c is a pointed map $(\mathbb{P}^1, \infty) \rightarrow (D_0(1), 0)$. It remains to show that the map

$$\begin{aligned} (\mathbb{P}^1)^{\wedge m} \wedge D_s(n) &\xrightarrow{\mathrm{id}^{\wedge m-1} \wedge c^* \wedge \mathrm{id}} (\mathbb{P}^1)^{\wedge m-1} \wedge D_s(1) \wedge D_s(n) \\ &\xrightarrow{(29)} (\mathbb{P}^1)^{\wedge m-1} \wedge D_s(m+1) \\ &\rightarrow \dots \\ &\rightarrow D_s(m+n) \end{aligned}$$

is a Σ_{m+n} -equivariant map of presheaves on \mathbf{Sm}/S , i.e., invariant under permuting the m wedge factors \mathbb{P}^1 . Given some map $f : U \rightarrow (\mathbb{P}^1)^{\times m}$ with $U \in \mathbf{Sm}/S$, let $f_i : U \rightarrow \mathbb{P}^1$ be the i -th projection of f and $c_i := f_i^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Given some form $\omega \in D(n)(U)$ (in some unspecified degree), the map is given by

$$(f, \omega) \mapsto c_1 \cdot_D (c_2 \cdot_D (\dots (c_m \cdot_D \omega) \dots)).$$

Here \cdot_D denotes the product on $D(*)$. The forms $c_i \in D_0(1)(U)$ are closed differential forms, so by Theorem 2.9(iv) the right hand expression is associative and commutative, i.e. invariant under the permutation action of Σ_m on $(\mathbb{P}^1)^{\times m}$.

By *loc. cit.*, the product on D is (graded) commutative and associative up to homotopy, thus the diagrams checking, say, the commutativity of $H_D \wedge H_D \rightarrow H_D$ do hold in $\mathbf{SH}(S)$. The details of that verification are omitted. \square

Remark 3.4. 1. The spectrum $\bigoplus_{p \in \mathbb{Z}} H_D\{p\}$ is given by replacing the p -th level $D_s(p)$ of H_D by D : it is enough to check that $\text{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, -)$ induces an isomorphism when applied to the map in question. By the compactness of $S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+$ in $\mathbf{SH}(S)$, this Hom-group commutes with the direct sum. Then the claim is trivial.

2. Choosing another metric λ on $\mathcal{O}(1)$ in the above definition, the resulting Deligne cohomology spectrum would be weakly equivalent to H_D since the difference of the Chern forms $c_1(\mathcal{O}(1), FS) - c_1(\mathcal{O}(1), \lambda)$ lies in the image of $d_D : D_1(1) \rightarrow D_0(1)$, see e.g. [Jos06, Lemma 5.6.1].

Lemma 3.5. *The Deligne cohomology spectrum H_D is an Ω -spectrum (with respect to smashing with \mathbb{P}^1).*

Proof: We have to check that the adjoint map to σ_p (Definition and Lemma 3.3),

$$b_p : D_s(p) \rightarrow \text{RHom}_\bullet(\mathbb{P}^1, D_s(p+1)),$$

is a motivic weak equivalence. As \mathbb{P}^1 is cofibrant and $D_s(p+1)$ is fibrant, the non-derived $\text{Hom}_\bullet(\mathbb{P}^1, D_s(p))$ is fibrant and agrees with $\text{RHom}_\bullet(\mathbb{P}^1, D_s(p))$. The map is actually a sectionwise weak equivalence, i.e., an isomorphism in $\mathbf{Ho}_{\text{sect}, \bullet}(S)$. To see this, it is enough to check that the map

$$D_s(p)(U) \rightarrow \text{Hom}_\bullet(\mathbb{P}^1, D_s(p+1)(U))$$

is a weak equivalence of simplicial sets for all $U \in \mathbf{Sm}/S$ [MV99, 1.8., 1.10, p. 50]. The m -th homotopy group of the left hand side is $H_D^{2p-m}(U, p)$ (Lemma 3.2), while π_m of the right hand simplicial set identifies with those elements of $\pi_m(\text{Hom}(\mathbb{P}^1 \times U, D_s(p+1))) = H_D^{2(p+1)-m}(\mathbb{P}^1 \times U, p+1)$ which restrict to zero when applying the restriction to the point $\infty \rightarrow \mathbb{P}^1$. By the projective bundle formula (25), the two terms agree. \square

Theorem 3.6. (i) *The ring spectrum H_D represents Deligne cohomology in $\mathbf{SH}(S)$: for any smooth scheme X over S , and any $n, m \in \mathbb{Z}$ we have*

$$\text{Hom}_{\mathbf{SH}(S)}((S^1)^{\wedge n} \wedge (\mathbb{P}_S^1)^{\wedge m} \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, H_D) = H_D^{n-2m}(X, -m).$$

(See p. 5 for the meaning of $(S^1)^{\wedge n}$, $(\mathbb{P}_S^1)^{\wedge m}$ with negative exponents.)

(ii) The Deligne cohomology spectrum H_D has a unique structure of an $H_{B,S}$ -algebra and $\oplus_{p \in \mathbb{Z}} H_D\{p\}$ has a unique structure of an BGL_S -algebra. In particular, H_D is an object in $\mathbf{DM}_B(S)$, so that (i) and (11) yield a natural isomorphism

$$\mathrm{Hom}_{\mathbf{DM}_B(S)}(M_S(X), H_D(p)[n]) = H_D^n(X, p)$$

for any smooth X/S .

(iii) The map $\mathrm{id}_D \wedge 1_{H_B} : H_D \rightarrow H_D \wedge H_B$ is an isomorphism in $\mathbf{SH}(S)_{\mathbb{Q}}$.

Definition 3.7. The maps induced by the unit of H_D are denoted $\rho_D : H_B \rightarrow H_D$ and $\mathrm{ch}_D : BGL \rightarrow \oplus_p H_D\{p\}$, respectively.

Proof: By Lemma 3.5, H_D is an Ω -spectrum. Thus (i) follows from Lemma 3.2.

(ii): by 3.3, H_D is a commutative ring spectrum. Recall the definition of étale descent for spectra and that for this it is sufficient that the individual pieces of the spectrum have étale descent [CD09, Def. 3.2.5, Cor. 3.2.18]. Thus, H_D satisfies étale descent by Theorem 2.9(vi). Any commutative ring spectrum in $\mathbf{SH}(S)_{\mathbb{Q}}$ satisfying étale descent admits a unique structure of an H_B -algebra [CD09, Corollary to Theorem 13, p. 7; 13.2.15]. This settles the claim for H_D . Secondly, the natural map (in $\mathbf{SH}(S)$)

$$BGL \rightarrow BGL_{\mathbb{Q}} \xrightarrow{(8)} \oplus_{p \in \mathbb{Z}} H_B\{p\} \xrightarrow{\rho_D\{p\}} \oplus_p H_D\{p\}$$

and the ring structure of $\oplus H_D\{p\}$ defines a BGL -algebra structure on $\oplus H_D\{p\}$. This uses that the isomorphism (8) is an isomorphism of monoid objects [CD09, 13.2.16]. The unicity of that structure follows from the unicity of the one on H_D and $\mathrm{Hom}_{\mathbf{SH}(S)}(BGL_{\mathbb{Q}}, \oplus H_D\{p\}) = \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(BGL_{\mathbb{Q}}, \oplus H_D\{p\})$, since H_D is a spectrum of \mathbb{R} - (a fortiori: \mathbb{Q} -)vector spaces.

(iii) follows from (ii), using again [CD09, 13.2.15]. \square

4 Arakelov motivic cohomology

Let S be a regular scheme of finite type over an arithmetic ring B . The generic fiber $S_{\eta} := S \times_{\mathbb{Z}} \mathbb{Q} \rightarrow B_{\eta} := B \times_{\mathbb{Z}} \mathbb{Q}$ is smooth, since B_{η} is a field (by Definition 2.7). We now define the Arakelov motivic cohomology spectrum $\widehat{H}_{B,S}$ which glues, in a sense, the Deligne cohomology spectrum $H_D \in \mathbf{SH}(S_{\eta})$ (Section 3) with the Beilinson motivic cohomology spectrum $H_{B,S}$ (7). Parallely, we do a similar construction with BGL_S instead of $H_{B,S}$. Once this is done, the framework of the stable homotopy category and motives readily imply the existence of functorial pullbacks and pushforwards for Arakelov motivic cohomology (Section 4.2). We also prove a higher arithmetic Riemann-Roch theorem (Theorem 4.15) and deduce further standard properties, such as the projective bundle formula in Section 4.4.

4.1 Definition

For any Noetherian scheme T , recall from Section 2.1 the category $\mathbf{Spt}(T) := \mathbf{Spt}^{\mathbb{P}^1}(\Delta^{\mathrm{op}}\mathbf{PSh}_{\bullet}(\mathbf{Sm}/T))$ with the stable model structure. The resulting homotopy category is $\mathbf{SH}(T)$. Let $\mathbf{Mon}(\mathbf{Spt}(T))$ be the category of (strictly) associative and unital monoid objects in $\mathbf{Spt}(T)$. It is endowed with a model structure such that the forgetful functor $\mathbf{Mon}(\mathbf{Spt}(T)) \rightarrow \mathbf{Spt}(T)$ is a Quillen right adjoint. Let $\mathcal{C}_{\mathbf{B}}$ be the full subcategory of $\mathbf{Mon}(\mathbf{Spt}(\mathrm{Spec}\mathbb{Z}))$ of objects which are isomorphic to $H_{\mathbf{B},\mathrm{Spec}\mathbb{Z}}$ in $\mathbf{SH}(\mathrm{Spec}\mathbb{Z})_{\mathbb{Q}}$. Similarly, let $\mathcal{C}_{\mathbf{BGL}}$ be the full subcategory of $\mathbf{Mon}(\mathbf{Spt}(\mathrm{Spec}\mathbb{Z}))$ consisting of objects that are isomorphic, in $\mathbf{SH}(\mathrm{Spec}\mathbb{Z})$, to $\mathbf{BGL}_{\mathrm{Spec}\mathbb{Z}}$. Recall that $\mathcal{C}_{\mathbf{BGL}}$ and $\mathcal{C}_{\mathbf{B}}$ are non-empty [CD09, 12.3.1, Cor. 13.2.6]. Let $f : S \rightarrow \mathrm{Spec}\mathbb{Z}$ be the structural map.

Definition 4.1. For any $A \in \mathbf{Mon}(\mathbf{Spt}(\mathrm{Spec}\mathbb{Z}))$, we put

$$\widehat{A} := \mathrm{hofib}_{\mathbf{Spt}(S)} \left(f^* A \xrightarrow{\mathrm{id} \wedge 1_D} f^* A \wedge \eta_* H_D \right) \in \mathbf{Spt}(S). \quad (31)$$

Here, hofib stands for the homotopy fiber, $1_D : S^0 \rightarrow H_D$ is the map corresponding to the unit of the DGA representing Deligne cohomology. Note that the map $\mathrm{id} \wedge 1_D$ is a map in $\mathbf{Spt}(S)$, as opposed to a mere map in $\mathbf{SH}(S)$.

We write $[\widehat{A}]$ for the image of \widehat{A} in $\mathbf{SH}(S)$ (or $\mathbf{SH}(S)_{\mathbb{Q}}$) under the localization functor. We put

$$\widehat{H_{\mathbf{B},S}} := \lim \left(F_{\mathbf{B}} : \mathcal{C}_{\mathbf{B}} \subset \mathbf{Mon}(\mathbf{Spt}) \rightarrow \mathbf{SH}(S)_{\mathbb{Q}}, F_{\mathbf{B}}(A) = [\widehat{A}] \right) \in \mathbf{SH}(S)_{\mathbb{Q}}$$

and

$$\widehat{BGL_S} := \lim \left(F_{\mathbf{BGL}} : \mathcal{C}_{\mathbf{BGL}} \subset \mathbf{Mon}(\mathbf{Spt}) \rightarrow \mathbf{SH}(S), F_{\mathbf{BGL}}(A) = [\widehat{A}] \right) \in \mathbf{SH}(S)$$

We call $\widehat{H_{\mathbf{B},S}}$ the *Arakelov motivic cohomology spectrum*.

Theorem 4.2. For any $A \in \mathcal{C}_{\mathbf{B}}$ and any $B \in \mathcal{C}_{\mathbf{BGL}}$, the natural maps

$$\widehat{H_{\mathbf{B},S}} \rightarrow [\widehat{A}], \quad \widehat{BGL_S} \rightarrow [\widehat{B}]$$

are isomorphisms in $\mathbf{SH}(S)_{\mathbb{Q}}$ and $\mathbf{SH}(S)$, respectively. The Chern character isomorphism $\mathrm{ch} : \mathbf{BGL}_{\mathrm{Spec}\mathbb{Z},\mathbb{Q}} \cong \bigoplus_{p \in \mathbb{Z}} H_{\mathbf{B},\mathrm{Spec}\mathbb{Z}}\{p\}$ gives rise to an isomorphism called Arakelov Chern character,

$$\widehat{\mathrm{ch}} : \widehat{BGL_{S,\mathbb{Q}}} \cong \bigoplus \widehat{H_{\mathbf{B},S}}\{p\}$$

in $\mathbf{SH}(S)_{\mathbb{Q}}$.

Proof: In $\mathbf{Ho}(\mathbf{Mon}(\mathbf{Spt}))$, any $A' \in \mathcal{C}_{\mathbf{B}}$ is isomorphic to A , and any two such isomorphisms are the same: by (6) and (11), we have

$$\mathrm{End}_{\mathbf{SH}(\mathrm{Spec}\mathbb{Z})_{\mathbb{Q}}}(H_{\mathbf{B},\mathrm{Spec}\mathbb{Z}}) = K_0(\mathrm{Spec}\mathbb{Z})_{\mathbb{Q}}^{(0)} = \mathbb{Q}.$$

Among these endomorphisms only the one corresponding to $1 \in \mathbb{Q}$ respects the unit of $\mathbf{H}_{\mathbf{B}, \text{Spec} \mathbb{Z}}$, so that

$$\text{End}_{\mathbf{Ho}(\mathbf{Mon}(\mathbf{Spt}))}(\mathbf{H}_{\mathbf{B}, \text{Spec} \mathbb{Z}}) = \{*\}.$$

The unique isomorphism $A' \rightarrow A$ (in $\mathbf{Ho}(\mathbf{Mon}(\mathbf{Spt}(\text{Spec} \mathbb{Z})))$) can be represented by a zig-zag of maps in $\mathbf{Mon}(\mathbf{Spt}(\text{Spec} \mathbb{Z}))$, which induces a zig-zag in $\mathbf{Spt}(S)$ between \widehat{A}' and \widehat{A} . The resulting isomorphism between $[\widehat{A}']$ and $[\widehat{A}]$ is independent of the choice of the zig-zag. In other words, the (non-full) subcategory of $\mathbf{SH}(S)_{\mathbb{Q}}$ consisting of objects and morphisms in the image of the functor $F_{\mathbf{B}}$ is equivalent to the category consisting of one object and one morphism. This yields the claim for $\widehat{\mathbf{H}_{\mathbf{B}, S}}$.

Likewise, any two zig-zags between B and another representative $B' \in \mathcal{C}_{\mathbf{BGL}}$ inducing the identity on $K_0(\text{Spec} \mathbb{Z}) = \mathbb{Z}$ give rise to the same isomorphism in $\mathbf{Ho}(\mathbf{Mon}(\text{Spec} \mathbb{Z}))$ (thus, in $\mathbf{SH}(\text{Spec} \mathbb{Z})$) [Rio07, Thm. IV.44].² In particular, since any zig-zag between B and B' induces a group homomorphism (with respect to addition) on $K_0(\text{Spec} \mathbb{Z})$, any zig-zag in the category of (multiplicative) monoids, $\mathbf{Mon}(\text{Spec} \mathbb{Z})$, induces the same map in $\mathbf{SH}(\text{Spec} \mathbb{Z})$. Now, the rest of the above argument carries over for $\widehat{\mathbf{BGL}_S}$.

For the Chern character we use the equivalence $\mathbf{SH}(S)_{\mathbb{Q}} \cong \mathbf{D}_{\mathbb{A}^1}(S, \mathbb{Q})$ (p. 5) in order to have rational coefficients on the model category, as opposed to the homotopy category only. After replacing domain and codomain of the Chern character isomorphism $\mathbf{BGL}_{\text{Spec} \mathbb{Z}, \mathbb{Q}} \xrightarrow{\cong} \oplus_p \mathbf{H}_{\mathbf{B}, \text{Spec} \mathbb{Z}}\{p\}$ by their fibrant-cofibrant replacements, the map can be represented by a map c of monoid objects in the model category underlying $\mathbf{D}_{\mathbb{A}^1}(\text{Spec} \mathbb{Z}, \mathbb{Q})$ [CD09, 13.2.16]. Now, $\text{hofib}(c \wedge (S^0 \xrightarrow{1_D} \mathbf{H}_D))$ yields a map of spectra $\widehat{c} : \widehat{\mathbf{BGL}}_{\mathbb{Q}, \text{Spec} \mathbb{Z}} \rightarrow \oplus_p \widehat{\mathbf{H}_{\mathbf{B}, \text{Spec} \mathbb{Z}}}\{p\}$. Any other representative c' of the Chern character is homotopic to c , hence so are the maps \widehat{c} and \widehat{c}' . That is, the resulting map $\widehat{c}\mathbf{h}$ is unique when passing to $\mathbf{D}_{\mathbb{A}^1}(\text{Spec} \mathbb{Z}, \mathbb{Q})$ and is an isomorphism there. \square

Remark 4.3. (i) The map in (31) is a map of A -modules. Taking the homotopy fiber in the category of A -modules yields an object $\widehat{A}^{\mathbf{Mod}} \in A\text{-}\mathbf{Mod}$. By the Quillen adjunction (11) and [Hir03, Theorem 19.4.5], $\widehat{A}^{\mathbf{Mod}}$ is weakly equivalent (in \mathbf{Spt}) to \widehat{A} . In this sense, it does not matter whether we take the homotopy fiber in \mathbf{Spt} or $A\text{-}\mathbf{Mod}$. In particular, $\widehat{\mathbf{H}_{\mathbf{B}, S}}$ is an object of $\mathbf{DM}_{\mathbf{B}}(S)$ and $\widehat{\mathbf{BGL}_S}$ is in $\mathbf{DM}_{\mathbf{BGL}}(S)$.

- (ii) We are mainly interested in gluing motivic cohomology with Deligne cohomology. However, nothing is special about Deligne cohomology. In fact, given some scheme $f : T \rightarrow S$ (not necessarily of finite type), and complexes of presheaves of \mathbb{Q} -vector spaces $D(p)$ on \mathbf{Sm}/T satisfying the conclusion of Theorem 2.9(ii), (iii), (v) (actually (25) suffices), (vi), and (iv), everything could be done with $f_*D(p)$ instead of $\eta_*D(p)$.

²Riou's theorem uses that $K_1(\text{Spec} \mathbb{Z}) = \mathbb{Z}^{\times} = \{\pm 1\}$ is torsion. The detour via $\text{Spec} \mathbb{Z}$ is not necessary for the construction of $\widehat{\mathbf{H}_{\mathbf{B}, S}}$, only for $\widehat{\mathbf{BGL}_S}$.

Definition 4.4. For any $M \in \mathbf{SH}(S)$, we define

$$\widehat{H}^n(M) := \mathrm{Hom}_{\mathbf{SH}(S)}(M, \widehat{\mathrm{BGL}}_S[n]),$$

$$\widehat{H}^n(M, p) := \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(M_{\mathbb{Q}}, \widehat{H}_{\mathbb{B}}(p)[n]).$$

The latter is called *Arakelov motivic cohomology* of M . For any finite type scheme $f : X \rightarrow S$, we define Arakelov motivic cohomology of X as

$$\widehat{H}^n(X/S, p) := \widehat{H}^n(M_S(X), p) = \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(f_! f^! H_{\mathbb{B}, S}, \widehat{H}_{\mathbb{B}, S}(p)[n])$$

and likewise

$$\widehat{H}^n(X/S) := \mathrm{Hom}_{\mathbf{DM}_{\mathrm{BGL}}(S)}(f_! f^! \mathrm{BGL}_S, \widehat{\mathrm{BGL}}_S[n])$$

When the base S is clear from the context, we will just write $\widehat{H}^n(X, p)$ and $\widehat{H}^n(X)$. See Theorem 4.15(i) for a statement concerning the independence of the base scheme S of the groups $\widehat{H}^n(X/S)$.

Theorem 4.5. (i) For any $M \in \mathbf{SH}(S)$ there are long exact sequences relating Arakelov motivic cohomology to (usual) motivic cohomology (12) and, for appropriate motives, Deligne cohomology (Definition 2.8):

$$\dots \rightarrow \widehat{H}^n(M, p) \rightarrow H^n(M, p) \xrightarrow{\rho} \mathrm{Hom}_{\mathbf{SH}(S)}(M, \eta_* H_{\mathbb{D}}(p)[n]) \rightarrow \widehat{H}^{n+1}(M, p) \dots \quad (32)$$

$$\dots \rightarrow \widehat{H}^n(M) \rightarrow H^n(M) \xrightarrow{\mathrm{ch}} \mathrm{Hom}_{\mathbf{SH}(S)}(M, \oplus \eta_* H_{\mathbb{D}}\{p\}[n]) \rightarrow \widehat{H}^{n+1}(M) \dots \quad (33)$$

The maps ρ and ch agree with the one induced by $\rho_{\mathbb{D}}$ and $\mathrm{ch}_{\mathbb{D}}$ (Definition 3.7).

(ii) For any l.c.i. scheme X/S (Definition 2.3, for example $X = S$) we get exact sequences

$$\dots \rightarrow \widehat{H}^n(X, p) \rightarrow K_{2p-n}(X)_{\mathbb{Q}}^{(p)} \rightarrow H_{\mathbb{D}}^n(X, p) \rightarrow \widehat{H}^{n+1}(X, p) \rightarrow \dots,$$

$$\dots \rightarrow \widehat{H}^n(X) \rightarrow K_{-n}(X) \rightarrow \oplus_p H_{\mathbb{D}}^{2p-n}(X, p) \rightarrow \widehat{H}^{n+1}(X) \rightarrow \dots \quad (34)$$

(iii) If $S' \xrightarrow{f} S$ is a scheme of positive characteristic over S , the obvious map $\widehat{H}^n(f_* M, p) \rightarrow H^n(f_* M, p)$ is an isomorphism for any $M \in \mathbf{SH}(S')$.

(iv) There are functorial isomorphisms $\widehat{H}^n(M, p) = \widehat{H}^n(M \otimes H_{\mathbb{B}, S}, p)$ and $\widehat{H}^n(M) = \widehat{H}^n(M \otimes \mathrm{BGL}_S)$. For any compact object M (such as $M = M_S(X)$, X/S of finite type), there is an isomorphism called the Arakelov Chern character:

$$\widehat{\mathrm{ch}} : \widehat{H}^n(M) \otimes_{\mathbb{Z}} \mathbb{Q} = \oplus_{p \in \mathbb{Z}} \widehat{H}^{n+2p}(M, p). \quad (35)$$

Proof: The long exact sequence in (i) follows from Theorem 3.6(iii) and generalities on the homotopy fiber in stable model categories. Similarly, $\mathrm{BGL} \wedge \mathrm{H}_\mathrm{D}$ is canonically isomorphic, via the Chern character, to $\oplus \mathrm{H}_\mathrm{B} \wedge \mathrm{H}_\mathrm{D}\{p\} \cong \oplus \mathrm{H}_\mathrm{D}\{p\}$. The agreement of ρ and ρ_D is also clear, since the H_B -module structure map $\mathrm{H}_\mathrm{B} \wedge \mathrm{H}_\mathrm{D} \rightarrow \mathrm{H}_\mathrm{D}$ is inverse to $1_\mathrm{B} \wedge \mathrm{id}_\mathrm{D} : \mathrm{H}_\mathrm{D} \rightarrow \mathrm{H}_\mathrm{B} \wedge \mathrm{H}_\mathrm{D}$.

For (ii), we apply (i) to $\mathrm{M}_S(X)$ and $f_! f^! \mathrm{BGL}_S$, respectively, where $f : X \rightarrow S$ is the structural map. In order to identify the motivic cohomology with the Adams eigenspace in K -theory, we use the adjunction (3) and the purity isomorphism for f (Example 2.4). To calculate $\mathrm{Hom}(f_! f^! \mathrm{H}_{\mathrm{B},S}, \mathrm{H}_\mathrm{D})$, we can replace B by the arithmetic field $B_\eta := B \times_{\mathbb{Z}} \mathbb{Q}$. The scheme S is regular, thus $s : S \rightarrow B$ is smooth (of finite type). The same is true for the structural map $x : X \rightarrow B$. Now, combining the relative purity isomorphisms for x and for s , we get an isomorphism

$$f^! \mathrm{H}_\mathrm{D} = f^! s^* \mathrm{H}_\mathrm{D} = f^! s^! \mathrm{H}_\mathrm{D}\{-\dim s\} = x^! \mathrm{H}_\mathrm{D}\{-\dim s\} = x^* \mathrm{H}_\mathrm{D}\{-\dim s + \dim x\} = f^* \mathrm{H}_\mathrm{D}\{\dim f\}.$$

We conclude

$$\begin{aligned} \mathrm{Hom}_{\mathbf{SH}(S)}(f_! f^! \mathrm{H}_{\mathrm{B},S}, \mathrm{H}_\mathrm{D}(p)[n]) &= \mathrm{Hom}(f^! \mathrm{H}_{\mathrm{B},S}, f^! \mathrm{H}_\mathrm{D}(p)[n]) \\ &= \mathrm{Hom}(f^* \mathrm{H}_{\mathrm{B},S}\{\dim f\}, f^* \mathrm{H}_\mathrm{D}(p)[n]\{\dim f\}) \\ &= \mathrm{Hom}(\mathrm{H}_{\mathrm{B},X}, \mathrm{H}_\mathrm{D}(p)[n]) \\ &\stackrel{3.6}{=} \mathrm{H}_\mathrm{D}^{2p-n}(X, p). \end{aligned}$$

(iii) follows from localization. The first two isomorphisms in (iv) follow from (11), (10), and Remark 4.3(i). The map $\widehat{\mathrm{ch}}$ is induced by the one in Theorem 4.2. \square

Remark 4.6. The arithmetic K_0 -groups $\widehat{K}_0(X)$ defined by Gillet and Soulé [GS90c, Section 6] for a regular projective variety X over a number ring \mathcal{O}_F behave differently than $\widehat{\mathrm{H}}^0(X/X)$ in that there is an exact sequence

$$K_1(X) \rightarrow \oplus_{p \geq 0} A^{p,p}(X) / (\mathrm{im} \partial + \mathrm{im} \overline{\partial}) \rightarrow \widehat{K}_0(X) \rightarrow K_0(X) \rightarrow 0,$$

where $A^{p,p}(X)$ is the group of real-valued (p,p) -forms ω on $X(\mathbb{C})$ such that $\mathrm{Fr}_\infty^* \omega = (-1)^p \omega$. In fact, we will establish elsewhere an isomorphism

$$\widehat{\mathrm{H}}^0(X/X) \cong \ker \left(\mathrm{ch} : \widehat{K}_0(X) \rightarrow \oplus_{p \geq 0} A^{p,p}(X) \right).$$

Similar remarks apply to Takeda's higher arithmetic K -groups and arithmetic Chow groups.

Remark 4.7. By (32), each group $\widehat{\mathrm{H}}^n(M)$ is an extension of a \mathbb{Z} -module by a quotient of a finite-dimensional \mathbb{R} -vector space by some \mathbb{Z} -module. Both \mathbb{Z} -modules are conjectured to be finitely generated in case $S = \mathrm{Spec} \mathbb{Z}$ and M compact (Bass conjecture). Similarly, the groups $\widehat{\mathrm{H}}^n(M, p)$ are extensions of \mathbb{Q} -vector spaces by groups of the form $\mathbb{R}^k / \text{some } \mathbb{Q}\text{-subspace}$. In particular,

we note that the Arakelov motivic cohomology groups $\hat{H}^n(M, p)$ are typically infinite-dimensional (as \mathbb{Q} -vector spaces). However, one can redo the above construction using the spectrum $H_B \otimes \mathbb{R}$ instead of H_B to obtain *Arakelov motivic cohomology groups with real coefficients*, $\hat{H}^n(M, \mathbb{R}(p))$. These groups are real vector spaces of conjecturally finite dimension, with formal properties similar to those of $\hat{H}^n(M, p)$, and these are the groups needed in the second author's conjecture on ζ and L -values [Sch10].

Example 4.8. We list the groups $\hat{H}^{-n} := \hat{H}^{-n}(\text{Spec } \mathcal{O}_F)$ of a number ring \mathcal{O}_F . These groups and their relation to the Dedekind ζ -function are well-known, cf. [Sou92, III.4], [Tak05, p. 623]. For any $n \in \mathbb{Z}$, (34) reads

$$H_D^0(X, n+1) \rightarrow \hat{H}^{-2n-1} \rightarrow K_{2n+1} \xrightarrow{\rho_*} H_D^1(X, n+1) \rightarrow \hat{H}^{-2n} \rightarrow K_{2n} \xrightarrow{\rho_*} H_D^0(X, n).$$

Elsewhere, we shall show that the map ρ_* induced by the BGL-module structure of $\oplus H_D\{p\}$ agrees with the Beilinson regulator, as it should. We conclude by Borel's work that \hat{H}^{-2n-1} is an extension of the torsion part of K_{2n+1} (which is μ_F for $n = 0$) by $H_D^0(X, 0)/\mathbb{Z}$ in case $n = 0$ and by $H_D^0(X, n+1)$ for $n \neq 0$. Moreover, for $n \neq 0$, \hat{H}^{-2n} is an extension of the torsion group $(K_{2n})_{\text{tor}}$ by a torus, i.e., a group of the form $\mathbb{R}^{s_n}/\mathbb{Z}^{s_n}$ for some s_n that can be read off (20). Finally \hat{H}^0 is an extension of the class group of F by a group $\mathbb{R}^{r_1+r_2-1}/\mathbb{Z}^{r_1+r_2-1} \oplus \mathbb{R}$.

For higher-dimensional varieties, the situation is less well-understood. For example, by Beilinson's, Bloch's, and Deninger's work we know that

$$K_{2n+2}(E)_{\mathbb{R}}^{(n+2)} \rightarrow H_D^2(E, n+2)$$

is surjective for $n \geq 0$, where E is a regular proper model of certain elliptic curves over a number field (for example a curve over \mathbb{Q} with complex multiplication in case $n = 0$). We refer to [Nek94, Section 8] for references and further examples.

4.2 Functoriality

We establish the expected functoriality properties of Arakelov motivic cohomology. Let $f : X \rightarrow Y$ be a map of S -schemes. The structural maps of X/S and Y/S are denoted x and y , respectively.

Lemma 4.9. *There is a functorial pullback*

$$f^* : \hat{H}^n(Y, p) \rightarrow \hat{H}^n(X, p), \quad f^* : \hat{H}^n(Y) \rightarrow \hat{H}^n(X).$$

More generally, for any map $\phi : M \rightarrow M'$ in $\mathbf{SH}(S)$ there is a functorial pullback

$$\phi^* : \hat{H}^n(M', p) \rightarrow \hat{H}^n(M, p), \quad \phi^* : \hat{H}^n(M') \rightarrow \hat{H}^n(M).$$

This pullback is compatible with the long exact sequence (32) and, for compact objects M and M' , with the Arakelov-Chern character (35).

Proof: The second statement is clear from the definition. The first claim follows by applying the natural transformation

$$x_! x^! = y_! f_! f^! y^! \xrightarrow{(3)} y_! y^!$$

to BGL_S or $\mathrm{H}_{\mathrm{B},S}$, respectively. The last statement is also clear since (3) is functorial, in particular it respects the isomorphism $\widehat{\mathrm{ch}} : \widehat{\mathrm{BGL}}_{\mathbb{Q},S} \cong \oplus \widehat{\mathrm{H}}_{\mathrm{B},S}\{p\}$. \square

In the remainder of this section, we assume that f and y (hence also x) is a regular projective map (Definition 2.3). Recall that $\dim f = \dim X - \dim Y$.

Definition and Lemma 4.10. *We define the pushforward*

$$f_* : \widehat{\mathrm{H}}^n(X, p) \rightarrow \widehat{\mathrm{H}}^{n-2\dim(f)}(Y, p - \dim(f))$$

on Arakelov motivic cohomology to be the map induced by the composition

$$\begin{aligned} \mathrm{M}_S(Y) = y_! y^! \mathrm{H}_{\mathrm{B},S} &\xrightarrow{(\mathrm{tr}_y^{\mathrm{B}})^{-1}} y_! y^* \mathrm{H}_{\mathrm{B},S}\{\dim(y)\} \\ &\xrightarrow{(2)} y_! f_* f^* y^* \mathrm{H}_{\mathrm{B},S}\{\dim(y)\} \\ &= x_! x^* \mathrm{H}_{\mathrm{B},S}\{\dim(y)\} \\ &\xrightarrow{\mathrm{tr}_x^{\mathrm{B}}} x_! x^! \mathrm{H}_{\mathrm{B},S}\{\dim(y) - \dim(x)\} \\ &= \mathrm{M}_S(X)\{-\dim(f)\}. \end{aligned}$$

Similarly,

$$f_* : \widehat{\mathrm{H}}^n(X) \rightarrow \widehat{\mathrm{H}}^n(Y)$$

is defined using the trace maps on BGL instead of the ones for H_{B} (15), (16).

This definition is functorial (with respect to the composition of regular projective maps).

Proof: Let $g : Y \rightarrow Z$ be a second map of S -schemes such that both g (hence $h := g \circ f$) and the structural map $z : Z \rightarrow S$ is regular projective. The functoriality of the pushforward is implied by the fact that the following two compositions agree (we do not write $\mathrm{H}_{\mathrm{B},-}\{-\}$ or BGL_- for space reasons):

$$\begin{aligned} z_! z^! &\xrightarrow{\mathrm{tr}_z^{-1}} z_! z^* \rightarrow z_! h_* h^* z^* = x_! x^* \xrightarrow{\mathrm{tr}_x} x_! x^! \\ z_! z^! &\xrightarrow{\mathrm{tr}_z^{-1}} z_! z^* \rightarrow z_! g_* g^* z^* = y_! y^* \xrightarrow{\mathrm{tr}_y} y_! y^! \xrightarrow{\mathrm{tr}_y^{-1}} y_! y^* \rightarrow y_! f_* f^* y^* = x_! x^* \xrightarrow{\mathrm{tr}_x} x_! x^! \end{aligned}$$

This agreement is an instance of the identity $ad_h = y_* ad_f y^* \circ ad_g$. \square

Proposition 4.11. *Let the following be a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S, \end{array}$$

where g is smooth and f (thus, f') is a regular projective map. Then, the base-change formula

$$g^* f_* = f'_* g'^* : \hat{H}^n(X) \rightarrow \hat{H}^n(Y')$$

and likewise for $\hat{H}^n(X, p) \rightarrow \hat{H}^{n-2 \dim f}(Y', p - \dim f)$ holds.

Proof: This is an immediate consequence of Proposition 2.5 and Remark 2.2. \square

Remark 4.12. The pushforward of arithmetic K -theory groups $\hat{K}_n(-)$ defined by Roessler ($n = 0$, [Roe99, Prop. 3.1]) and Takeda ($n \geq 0$, [Tak05, Section 7.3.]) applies to smooth projective maps $f : X \rightarrow Y$ between arithmetic varieties (flat over \mathbb{Z} and regular). A functoriality statement for analytic torsion, which constitute the technically most challenging part of this pushforward, has been given by Faltings and Ma [Fal92, Theorem 5.5.], [Ma99, (0.5)]. These pushforwards need an auxiliary choice of a metric on the relative tangent bundle. The pushforward on arithmetic Chow groups [GS90a, Theorem 3.6.1] is defined for all proper maps between arithmetic varieties. For the time being, no pushforward has been established for the higher arithmetic Chow groups [BGF].

4.3 Purity and an arithmetic Riemann-Roch theorem

In this subsection, we establish a purity isomorphism and a Riemann-Roch theorem for Arakelov motivic cohomology. We cannot prove it in the expected full generality of regular projective maps, but need some smoothness assumption.

Given any closed immersion $i : Z \rightarrow \operatorname{Spec} \mathbb{Z}$, we let $j : U \rightarrow \operatorname{Spec} \mathbb{Z}$ be its open complement. The generic point is denoted $\eta : \operatorname{Spec} \mathbb{Q} \rightarrow \operatorname{Spec} \mathbb{Z}$. We also write i, j, η for the pullback of these maps to any scheme, e.g., $i : X_Z := X \times_{\operatorname{Spec} \mathbb{Z}} Z \rightarrow X$. Recall that B is an arithmetic ring whose generic fiber B_η is a field (Definition 2.7).

Let $f : X \rightarrow S$ be a map of regular B -schemes. For clarity, we write $D(p)_{X_\eta}$ for the complex of presheaves on \mathbf{Sm}/X_η that was denoted $D(p)$ above and H_{D, X_η} for the resulting spectrum. Moreover, we write $H_{D, X} := \eta_* H_{D, X_\eta} \in \mathbf{SH}(X)$. The complex $D(p)_{X_\eta}$ is the restriction of the complex $D(p)_{B_\eta}$. Therefore, there is a natural map $f^* D(p)_S \rightarrow D(p)_X$, which in turn gives rise to a map of spectra

$$\alpha_D^f : f^* H_{D, S} \rightarrow H_{D, X}.$$

This map is an isomorphism if f is smooth, since $f^* : \mathbf{PSh}(\mathbf{Sm}/S) \rightarrow \mathbf{PSh}(\mathbf{Sm}/X)$ is just the restriction in this case. Is α_D^f an isomorphism for a closed immersion f between flat regular B -schemes? The corresponding fact for BGL, i.e., the isomorphism $f^* \operatorname{BGL}_S = \operatorname{BGL}_X$ ultimately relies on the fact that algebraic K -theory of smooth schemes over S is represented in $\mathbf{SH}(S)$ by the infinite Grassmannian, which is a smooth scheme over S . Therefore, it would be interesting to have a geometric description of the spectrum representing Deligne cohomology, as opposed to the merely cohomological representation given by the complexes $D(p)$.

Lemma 4.13. (i) Given another map $g : Y \rightarrow X$ of regular B -schemes, there is a natural isomorphism of functors $\alpha_D^g \circ g^* \alpha_D^f = \alpha_D^{f \circ g}$.

(ii) The following are equivalent

- α_D^f is an isomorphism in $\mathbf{SH}(X)$.
- For any $i : Z \rightarrow \mathrm{Spec} \mathbb{Z}$, the object $i^! f^* H_{D,S}$ is zero in $\mathbf{SH}(X \times_{\mathrm{Spec} \mathbb{Z}} Z)$.
- For any sufficiently small $j : U \rightarrow \mathrm{Spec} \mathbb{Z}$, the adjunction morphism $f^* H_{D,S} \rightarrow j_* j^* f^* H_{D,S}$ is an isomorphism in $\mathbf{SH}(X)$.

(iii) The conditions in (ii) are satisfied if f fits into a diagram

$$\begin{array}{ccccc} X & \xrightarrow{x} & B' & \longrightarrow & B \\ f \downarrow & \nearrow s & & & \\ S & & & & \end{array}$$

where B' is regular and of finite type over B , x and s are smooth. In particular, this applies when f is smooth or when both X and S are smooth over B .

Proof: (i) is easy to verify using the definition of the pullback functor. (iii) is a consequence of the above remark and (i) using the chain of natural isomorphisms $f^* H_{D,S} = f^* s^* H_{D,B'} = x^* H_{D,B'} = H_{D,X}$. For (ii), consider the map of distinguished localization triangles

$$\begin{array}{ccccc} i_* i^! f^* H_{D,S} & \longrightarrow & f^* H_{D,S} & \longrightarrow & j_* j^* f^* H_{D,S} \\ \downarrow & & \downarrow \alpha_D^f & & \downarrow j_* j^* \alpha_D^f = j_* \alpha_D^{f_U} \\ 0 = i_* i^! H_{D,X} & \longrightarrow & H_{D,X} & \longrightarrow & j_* j^* H_{D,X}. \end{array}$$

The map $\alpha_D^{f_U}$ is an isomorphism as soon as j is small enough so that X_U and S_U are smooth over B_U . Such a j exists by the regularity of X and S . This shows the equivalence of the three statements in (ii). \square

Below, we write $\mathbb{B} := \bigoplus_{p \in \mathbb{Z}} H_{\mathbb{B}}\{p\}$ and $\widehat{\mathbb{B}}_X := \mathrm{hofib}(\mathbb{B}_X \rightarrow \mathbb{B}_X \wedge H_{D,X})$.

Theorem 4.14. Let $f : X \rightarrow S$ be a regular projective map (Definition 2.3). Then there is a commutative diagram in $\mathbf{SH}(X)_{\mathbb{Q}}$:

$$\begin{array}{ccccccc} \widehat{\mathrm{BGL}}_X & \xleftarrow{\widehat{\alpha}} & f^* \widehat{\mathrm{BGL}}_S & \xrightarrow{\widehat{\mathrm{tr}}_{\mathrm{BGL}}} & f^? \widehat{\mathrm{BGL}}_S & \xrightarrow{\beta} & f^! \widehat{\mathrm{BGL}}_S \\ \downarrow \widehat{\mathrm{ch}}_X & & \downarrow f^* \widehat{\mathrm{ch}}_S & & \downarrow f^? \widehat{\mathrm{ch}}_S & & \downarrow f^! \widehat{\mathrm{ch}}_S \\ \widehat{\mathbb{B}}_X & \xleftarrow{\widehat{\alpha}} & f^* \widehat{\mathbb{B}}_S & \xrightarrow{\widehat{\mathrm{Td}}(T_f)} & f^? \widehat{\mathbb{B}}_S & \xrightarrow{\widehat{\mathrm{tr}}_{\mathbb{B}}} & f^! \widehat{\mathbb{B}}_S \end{array} \quad (36)$$

All vertical morphisms and the maps in the middle quadrangle are isomorphisms (in $\mathbf{SH}(X)_{\mathbb{Q}}$). Moreover, the horizontal maps in the outer squares are isomorphisms provided α_D^f is an isomorphism. In particular (Lemma 4.13 (iii)) this applies when B is a field or when X and S are smooth over B or when f is smooth.

Proof: By Theorem 4.2, we can choose cofibrant-fibrant representatives for \mathbf{BGL} and \mathbf{H}_B . To define the maps $\widehat{\alpha}$ in (36), consider the commutative diagram of spectra

$$\begin{array}{ccc} f^*\mathbf{BGL}_S & \xrightarrow{\alpha_{\mathbf{BGL}}^f} & \mathbf{BGL}_X \\ \downarrow f^*(\mathrm{id} \wedge 1_D) & & \downarrow \mathrm{id} \wedge 1_D \\ f^*(\mathbf{BGL}_S \wedge \mathbf{H}_{D,S}) & \xrightarrow{\cong} f^*\mathbf{BGL}_S \wedge f^*\mathbf{H}_{D,S} \xrightarrow{\alpha_{\mathbf{BGL}}^f \wedge \alpha_D^f} & \mathbf{BGL}_X \wedge \mathbf{H}_{D,X}. \end{array}$$

Here $\alpha_{\mathbf{BGL}}^f : f^*\mathbf{BGL}_S \rightarrow \mathbf{BGL}_X$ is, by definition of \mathbf{BGL} , an isomorphism of spectra. The lower left hand map expresses the fact that f^* is a monoidal functor. The diagram commutes because of $\alpha_D^f(f^*1_D) = 1_D$. This diagram induces a map between the homotopy fibers of the two vertical maps, which are $f^*\widehat{\mathbf{BGL}}_S$ and $\widehat{\mathbf{BGL}}_X$, respectively. This is the map $\widehat{\alpha}$ above. The one for $\widehat{\mathbf{B}}$ is constructed the same way by replacing \mathbf{BGL} by \mathbf{B} throughout. Using $f^*\mathrm{ch}_S = \mathrm{ch}_X$, this shows the commutativity of the left hand square in (36). Of course, both maps $\widehat{\alpha}$ are isomorphisms in $\mathbf{SH}(X)$ when α_D^f is so.

We define

$$f^? \widehat{\mathbf{BGL}}_S := \mathrm{hofib}(f^! \mathbf{BGL}_S \xrightarrow{\mathrm{id} \wedge 1} f^! \mathbf{BGL}_S \wedge f^* \mathbf{H}_{D,S})$$

and similarly for $f^? \widehat{\mathbf{B}}_S$. (The notation is not meant to suggest a functor $f^?$, it is just a shorthand.) The map $f^? \widehat{\mathrm{ch}}$ is induced by $\mathrm{ch} : \mathbf{BGL}_S \rightarrow \mathbf{B}_S$. The horizontal maps in the middle quadrangle are defined in a similar manner: for example, consider the map $\mathrm{tr}_{\mathbf{BGL}} : f^* \mathbf{BGL} \rightarrow f^! \mathbf{BGL}$. This map is only defined as a map in the homotopy category $\mathbf{SH}(X)$. Replacing domain and codomain of $\mathrm{tr}_{\mathbf{BGL}}$ by their cofibrant-fibrant replacements, respectively, we can assume that $\mathrm{tr}_{\mathbf{BGL}}$ is represented by an actual map $t : f^* \mathbf{BGL} \rightarrow f^! \mathbf{BGL}$ of spectra. Now, put $\widehat{t} := \mathrm{hofib}(t \wedge (S^0 \xrightarrow{1_D} f^* D))$. Any two maps t and t' representing $\mathrm{tr}_{\mathbf{BGL}}$ are homotopic to one another. Hence so are \widehat{t} and \widehat{t}' , so they represent the same map in $\mathbf{SH}(X)$. This map is denoted $\widehat{\mathrm{tr}}_{\mathbf{BGL}}$. Similarly, we define $\widehat{\mathrm{Td}}(T_f)$ (viewing $\mathrm{Td}(T_f)$ as a map $f^* \mathbf{B}_S \rightarrow f^* \mathbf{B}_S$) and $\widehat{\mathrm{tr}}_{\mathbf{B}}$. Picking representatives of all maps, the quadrangle will in general not commute in the category of spectra, but does so up to homotopy, by construction and by the Riemann-Roch theorem 2.6. This settles the middle rectangle.

As for the rightmost square, we define $\beta : f^? \widehat{\mathbf{BGL}}_S \rightarrow f^! \widehat{\mathbf{BGL}}_S$ to be the

map between the homotopy fibers of the vertical maps in the following diagram:

$$\begin{array}{ccc} f^! \mathrm{BGL}_S & \xlongequal{\quad} & f^! \mathrm{BGL}_S \\ \downarrow \mathrm{id} \wedge 1_D & & \downarrow f^!(\mathrm{id} \wedge 1_D) \\ f^! \mathrm{BGL}_S \wedge f^* \mathrm{H}_{D,S} & \xrightarrow{\gamma} & f^!(\mathrm{BGL}_S \wedge \mathrm{H}_{D,S}) \end{array}$$

Here γ is adjoint to the natural map (obtained by the projection formula [CD09, 2.4.21(v)] and the adjunction counit),

$$f_!(f^! \mathrm{BGL}_S \wedge f^* \mathrm{H}_{D,S}) = f_! f^! \mathrm{BGL}_S \wedge \mathrm{H}_{D,S} \rightarrow \mathrm{BGL}_S \wedge \mathrm{H}_{D,S}.$$

The commutativity of the diagram (in the category of spectra) is an immediate consequence of the functoriality of the projection formula isomorphism applied to the map $1_D : S^0 \rightarrow \mathrm{H}_{D,S}$. Again, the same recipe applies to $f^? \widehat{\mathrm{B}}_S \rightarrow f^! \widehat{\mathrm{B}}_S$ and also shows the commutativity of the rightmost square in (36).

Now, suppose α_D^f is an isomorphism. This implies that $\alpha_D^{f_U}$ is an isomorphism as well, for any U . By the regularity of X and S , we can choose j such that X_U and S_U are smooth over B_U . We let $i : Z = (\mathrm{Spec} \mathbb{Z} \setminus U)_{\mathrm{red}} \rightarrow \mathrm{Spec} \mathbb{Z}$ be the complement of j . We show that γ and thus β is an isomorphism (in $\mathbf{SH}(X)$). Consider the diagram in $\mathbf{SH}(X)$,

$$\begin{array}{ccc} f^! \mathrm{BGL}_S \wedge f^* \mathrm{H}_{D,S} & \xrightarrow{\gamma} & f^!(\mathrm{BGL}_S \wedge \mathrm{H}_{D,S}) \\ \downarrow \text{by Lemma 4.13(ii)} \cong & & \downarrow \cong \\ f^! \mathrm{BGL}_S \wedge j_* j^* f^* \mathrm{H}_{D,S} & \xrightarrow{\gamma'} & f^!(\mathrm{BGL}_S \wedge j_* j^* \mathrm{H}_{D,S}), \end{array}$$

where γ' is the map making the diagram commutative. It suffices to show that $i^! \gamma'$ and $j^* \gamma'$ are isomorphisms. In $\mathbf{SH}(X_Z)$, there is an isomorphism

$$i^!(f^! \mathrm{BGL}_S \wedge j_* j^* f^* \mathrm{H}_{D,S}) \cong i^!(\mathrm{BGL}_X \wedge j_* \mathrm{H}_{D,X_U}) \cong i^!(\mathrm{BGL}_X \wedge \mathrm{H}_{D,X}) \cong i^! \oplus_{p \in \mathbb{Z}} \mathrm{H}_{D,X}\{p\} = 0.$$

We have used the absolute purity isomorphism for BGL (applied to the regular projective map f , cf. Example 2.4) and 3.6(iii). By the same token,

$$i^! f^!(\mathrm{BGL}_S \wedge j_* j^* \mathrm{H}_{D,S}) \cong i^! f^! \oplus_{p \in \mathbb{Z}} \mathrm{H}_{D,S}\{p\} = 0.$$

On the other hand, using $j^* = j^!$, we see that $j^* \gamma'$ is the canonical map

$$f_U^! \mathrm{BGL}_{S_U} \wedge \mathrm{H}_{D,X_U} \xrightarrow{4.13(iii)} f_U^! \mathrm{BGL}_{S_U} \wedge j^* f^* \mathrm{H}_{D,S} \rightarrow f_U^!(\mathrm{BGL}_{S_U} \wedge \mathrm{H}_{D,S_U})$$

obtained in the same way as γ . We have the following situation:

$$\begin{array}{ccc} X_U & \xrightarrow{a} & S_U \\ & \searrow & \downarrow f_U \\ & & B_U \end{array}$$

$\mathbb{P}_{S_U}^{D^n}$

where a is a closed immersion and p and every map in the diagram with codomain B_U is smooth. Therefore, $f_U^! M$ is functorially isomorphic to $f_U^* M\{\dim f_U\}$ and $j^* \gamma'$ is an isomorphism for any $M \in \mathbf{SH}(S_U)$ by construction of the relative purity isomorphism by Ayoub [Ayo07, Section 1.6]. \square

We can now conclude a higher arithmetic Riemann-Roch theorem. It controls the failure of $\widehat{\text{ch}}$ to commute with the pushforward.

Theorem 4.15. *Let $f : X \rightarrow S$ be a regular projective map (Definition 2.3) of schemes of finite type over an arithmetic ring B (Definition 2.7). Moreover, we assume that f is such that*

$$\alpha_D^f : f^* H_{D,S} \rightarrow H_{D,X}$$

is an isomorphism. This condition is satisfied, for example, when f is smooth or when X and S are smooth over B (Lemma 4.13). Then, the following holds:

- (i) (Purity) *The absolute purity isomorphisms for BGL and H_B (14) induce isomorphisms*

$$\widehat{\text{BGL}}_X \cong f^* \widehat{\text{BGL}}_S \cong f^! \widehat{\text{BGL}}_S, \quad \widehat{H}_B X \cong f^* \widehat{H}_B S \cong f^! \widehat{H}_B S\{-\dim f\}.$$

In particular, Arakelov motivic cohomology is independent of the base scheme in the sense that there are isomorphisms

$$\widehat{H}^n(X/S) \cong \widehat{H}^n(X/X), \quad \widehat{H}^n(X/S, p) \cong \widehat{H}^n(X/X, p).$$

- (ii) (Higher arithmetic Riemann-Roch theorem) *There is a commutative diagram*

$$\begin{array}{ccc} \widehat{H}^n(X/X) & \xrightarrow{f_*} & \widehat{H}^n(S/S) \\ \downarrow \widehat{\text{ch}}_X & & \downarrow \widehat{\text{ch}}_S \\ \bigoplus_{p \in \mathbb{Z}} \widehat{H}^{n+2p}(X, p) & \xrightarrow{f_* \circ \widehat{\text{Td}}(T_f)} & \bigoplus_{p \in \mathbb{Z}} \widehat{H}^{n+2p}(S, p). \end{array}$$

Here, the top line map f_ is given by*

$$\begin{aligned} \widehat{H}^n(X/X) &:= \text{Hom}_{\mathbf{SH}(X)}(S^{-n}, \widehat{\text{BGL}}_X) \\ &\xrightarrow{(36)} \text{Hom}_{\mathbf{SH}(X)}(S^{-n}, f^! \widehat{\text{BGL}}_S) \\ &\xrightarrow{(2)} \text{Hom}_{\mathbf{SH}(S)}(S^{-n}, \widehat{\text{BGL}}_S) = \widehat{H}^n(S/S). \end{aligned}$$

Using the identifications $\widehat{H}^n(X/X) \cong \widehat{H}^n(X/S)$, this map agrees with the one defined in 4.10. The bottom map f_ is given similarly replacing BGL with B .*

Proof: The isomorphisms for $\widehat{\text{BGL}}_?$ in (i) are a restatement of Theorem 4.14. The ones for $\widehat{\text{H}}_{\text{B}}?$ also follow from that by dropping the isomorphism $\widehat{\text{Td}}(\widehat{T}_f)$ in the bottom row of (36) and noting that tr_{B} , hence $\widehat{\text{tr}}_{\text{B}}$ shifts the degree by $\dim f$. The isomorphisms in the second statement are given by the following identifications of morphisms in $\mathbf{DM}_{\text{BGL}}(-)$:

$$\begin{aligned} \text{Hom}(\text{BGL}_X, \widehat{\text{BGL}}_X) &\xrightarrow{4.14} \text{Hom}(\text{BGL}_X, f^! \widehat{\text{BGL}}_S) \\ &\xrightarrow{(\text{tr}_{\text{BGL}})^{-1}} \text{Hom}(f^! \text{BGL}_S, f^! \widehat{\text{BGL}}_S) = \text{Hom}(f_! f^! \text{BGL}_S, \widehat{\text{BGL}}_S). \end{aligned}$$

and likewise for H_{B} .

(ii) is an immediate corollary of Theorem 4.14, given that the two isomorphisms (in $\mathbf{SH}(X)_{\mathbb{Q}}$) $\widehat{\text{Td}}(\widehat{T}_f) \circ \widehat{\alpha}^{-1}$ and $\widehat{\alpha}^{-1} \circ \widehat{\text{Td}}(\widehat{T}_f)$, where $\text{Td}(\widehat{T}_f)$ is seen as an endomorphism of $f^* \text{B}_S$ and of B_X , respectively, agree. This agreement follows from the definition of $\widehat{\alpha}$. The agreement of the two definitions of f_* is clear from the definition. \square

This also elucidates the behavior of (34) with respect to pushforward: in the situation of the theorem, the pushforward $f_* : \widehat{\text{H}}^n(X) \rightarrow \widehat{\text{H}}^n(S)$ sits between the usual K -theoretic pushforward and the pushforward on Deligne cohomology (which is given by integration of differential forms along the fibers in case $f(\mathbb{C})$ is smooth, and by pushing down currents in general), multiplied with the Todd class (in Deligne cohomology) of the relative tangent bundle.

4.4 Further properties

Theorem 4.16. (i) *Arakelov motivic cohomology satisfies h -descent (thus, a fortiori, Nisnevich, étale, cdh, qfh and proper descent). For example, there is an exact sequence*

$$\dots \rightarrow \widehat{\text{H}}^n(X, p) \rightarrow \widehat{\text{H}}^n(U \sqcup V, p) \rightarrow \widehat{\text{H}}^n(W, p) \rightarrow \widehat{\text{H}}^{n+1}(X, p) \rightarrow \dots$$

where

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{f} & X \end{array}$$

is a cartesian square of smooth schemes over S that is either a distinguished square for the cdh-topology (f a closed immersion, p is proper such that p is an isomorphism over $X \setminus U$), or a distinguished square for the Nisnevich topology (f an open immersion, p étale inducing an isomorphism $(p^{-1}(X \setminus U))_{\text{red}} \rightarrow (X \setminus U)_{\text{red}}$), or such that $U \sqcup V \rightarrow X$ is an open cover.

(ii) *Arakelov motivic cohomology is homotopy invariant and satisfies a projective bundle formula:*

$$\widehat{\text{H}}^n(X \times \mathbb{A}^1, p) \cong \widehat{\text{H}}^n(X, p),$$

$$\widehat{H}^n(\mathbf{P}(E), p) \cong \oplus_{i=0}^d \widehat{H}^{n-2i}(X, p-i).$$

Here X/S is arbitrary (of finite type), $E \rightarrow X$ is a vector bundle of rank $d+1$, $\mathbf{P}(E)$ is its projectivization.

- (iii) Any distinguished triangle of motives induces long exact sequences of Arakelov motivic cohomology. For example, let X/S be an l.c.i. scheme (Example 2.4). Let $i : Z \subset X$ be a closed immersion of regular schemes of constant codimension c with open complement $j : U \subset X$. Then there is an exact sequence

$$\widehat{H}^{n-2c}(Z, p-c) \xrightarrow{i_*} \widehat{H}^n(X, p) \xrightarrow{j^*} \widehat{H}^n(U, p) \rightarrow \widehat{H}^{n+1-2c}(Z, p-c).$$

- (iv) The cdh-descent and the properties (ii), (iii) hold *mutatis mutandis* for $\widehat{H}^*(-)$.

Proof: The h-descent is a general property of modules over $H_{B,S}$ [CD09, Thm 15.1.3]. The \mathbb{A}^1 -invariance and the bundle formula are immediate from $M(X) \cong M(X \times \mathbb{A}^1)$ and $M(\mathbf{P}(E)) \cong \oplus_{i=0}^d M(X)\{i\}$. For the last statement, we use the localization exact triangle [CD09, 2.3.5] for $U \xrightarrow{j} X \xleftarrow{i} Z$:

$$f_! j_! j^! f^! H_{B,S} \rightarrow f_! f^! H_{B,S} \rightarrow f_! i_* i^* f^! H_{B,S}.$$

The purity isomorphism $f^* H_{B,S}\{\dim f\} = f^! H_{B,S}$ (Example 2.4) for the structural map $f : X \rightarrow S$ and the absolute purity isomorphism (14) for i imply that the rightmost term is isomorphic to $f_! i_* i^! f^! H_{B,S}\{-\dim i\} = M_S(Z)\{-c\}$. Mapping this triangle into $\widehat{H}_{B,S}(p)[n]$ gives the desired long exact sequence.

The arguments for \widehat{BGL}_S are the same. The only difference is that descent for topologies exceeding the cdh-topology requires rational coefficients. \square

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